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Classical solutions of the TEK model and noncommutative instantons in two dimensions

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Abstract: The twisted Eguchi-Kawai (TEK) model provides a non-perturbative definition of noncommutative Yang-Mills theory: the continuum limit is approached at large-N by performing suitable double scaling limits, in which non-planar contributions are no longer suppressed. We consider here the two-dimensional case, trying to recover within this framework the exact results recently obtained by means of Morita equivalence. We present a rather explicit construction of classical gauge theories on noncommutative toroidal lattice for general topological charges. After discussing the limiting procedures to recover the theory on the noncommutative torus and on the noncommutative plane, we focus our attention on the classical solutions of the related TEK models. We solve the equations of motion and we find the configurations having finite action in the relevant double scaling limits. They can be explicitly described in terms of twist-eaters and they exactly correspond to the instanton solutions that are seen to dominate the partition function on the noncommutative torus. Fluxons on the noncommutative plane are recovered as well. We also discuss how the highly non-trivial structure of the exact partition function can emerge from a direct matrix model computation. The quantum consistency of the TEK formulation is eventually checked by computing Wilson loops in a particular limit.

Keywords: Solitons Monopoles and Instantons, 1/N Expansion, Non-Commutative Geometry, Matrix Models.

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Contents

1. Introduction

The possibility to embed consistently a noncommutative field theory into a string theory [1, 2] has stimulated in the last years a large amount of studies, trying to understand classical and quantum noncommutative dynamics both at perturbative and non-perturbative level (see [3, 4] for a review). In particular quantum field theories on noncommutative spacetimes represent a framework in which to study D-branes physics retaining part of the non-locality inherent in string theory. From a purely field theoretical point of view, instead, they appear as a highly non-trivial non-local deformation of conventional quantum field theory, presenting a large variety of new phenomena not completely understood even in the basic cases: at perturbative level the UV/IR mixing [5, 6] complicates the renormalization program and it may produce tachyonic instabilities [7]. In both cases an intriguing interplay between perturbative and non-perturbative effects seems to conspire in order to recover a consistent physical picture: in [8] it was shown that resummation of perturbation theory is mandatory to have a sensible infrared limit in four-dimensional scalar theories while the results of [9] seems to indicate that the presence of new (extended) degrees of freedom may overcome the instability problem. The possibility of transitions to new "exotic" phases has also been put forward [10].

On the other hand the analysis presented in [11] clearly suggests that UV/IR mixing is not a perturbative artefact, being intimately related to the non-perturbative structure of noncommutative gauge theories. Lattice regularization with periodic boundary conditions is in fact equivalent, in this case, to a well-known unitary (multi)matrix model, the twisted Eguchi-Kawai (TEK) model [12]: in the usual large-N limit it is believed to reproduce Yang-Mills theory in the 't Hooft regime (see [13] for a review on the subject). The novelty of its noncommutative incarnation is that to reach the continuum limit one has to perform a *double scaling* limit, in which the dimensionful lattice spacing scales to zero with a precise power of N: non-planar contributions are no longer suppressed and carry the relevant physics of the noncommutative theory.

The possibility to have an explicit non-perturbative formulation of noncommutative gauge theories through matrix models is not unexpected of course: noncommutative Yang-Mills theory was directly obtained from the large-N limit of the IIB matrix model [14]. The spacetime dependence emerged from expanding around a classical vacuum, but initially it is hidden in the infinitely many degrees of freedom of the (large) matrices: on the other hand the original appearance of noncommutative geometry from string theory was based on the very same mechanism [1].

A related and complementary approach to non-perturbative physics of noncommutative gauge theories was advocated in [15, 16, 17], by using Morita equivalence [18]. The basic idea is to start from the theory compactified on a rational noncommutative torus T^D (by rational we mean that the entries on the antisymmetric matrix $\theta_{\mu\nu}$ describing the noncommutative star-product are rational numbers when rescaled with the torus radii) and to reach noncommutative $\mathbb{R}^D$ by a suitable decompactification limit. By Morita equivalence the theory on a rational noncommutative torus is equivalent to some commutative theory with 't Hooft fluxes. Then the infinite space is attained by performing a large-N limit that simultaneously shrinks the size of the (commutative) torus to zero with a specific powers of N. Again a double scaling limit is required, confirming the non-perturbative nature of UV/IR mixing. Particularly explicit results have been obtained in the simplified context of two-dimensional noncommutative gauge theories. In [17], employing the above strategy and the known solution of the Morita equivalent theory [19], we derived the partition function and the correlators of two Wilson lines on the noncommutative plane. A remarkable fact was that the classical configurations, naturally dominating this limit, are in correspondence with the noncommutative fluxons found in [20, 21, 22]. This result also establishes an unexpected relation between the double scaling limit and classical solutions of the (noncommutative) equations of motion. In ordinary two-dimensional Yang-Mills theory, this connection is instead absent: in fact no classical configuration survives the usual large-N limit.

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Later in ref. [23] a localization theorem for Yang-Mills theory on the noncommutative two-torus was proven, generalizing the Witten's result [24] on the exactness of semiclassical approximation for familiar two-dimensional gauge theories on Riemann surfaces. This beautiful result led the authors to propose an interesting formula for the exact partition function on the noncommutative torus, consistent with ours in the decompactification limit. They were able to write the partition function, in the rational case, as an expansion around the critical points of Yang-Mills theory on a particular projective module, exploiting the mentioned equivalence with the parent commutative theory and the Poisson-resummed formulae presented in [25]. Assuming smoothness on the noncommutative parameter, they obtained an intriguing generalization to the irrational case: an important consistency check with the localization formula was the peculiar form of the contribution of the quantum fluctuations, reflecting the singularity structure of the moduli space of constant curvature connections [26]. Recently the same authors extended their analysis to compute Wilson line correlation functions [27] and they offered some evidences for relating the noncommutative theory to ordinary (commutative) generalized YM_2 [28].

The derivation of the above results (both ours and the ones contained in [23, 27]), although giving useful insights on the non-perturbative structure of quantum noncommutative theories, may be considered, in some sense, unsatisfactory. In fact it heavily relies on the equivalence between theories with *rational* noncommutative parameter and ordinary Yang-Mills theories and it takes advantage of the exact solution of the latter in two dimensions (solution that is not available in higher dimensions of course). On the other hand, as we mentioned before, a general non-perturbative formulation based on the double scaling limit of TEK model exists and can be used in any dimension: it would be nice to recover the results of [17, 23, 27] starting from this very fundamental definition of noncommutative Yang-Mills theory by exploiting familiar matrix model/lattice techniques. Incidentally TEK model in two dimensions was the subject of intense studies in the eighties [29], in order to better understand its claimed equivalence with conventional Wilson approach to lattice gauge theories at large N. At the time, of course, no interest in searching a non-trivial double scaling limit has been raised, although later similar investigations on two-dimensional Wilson theory [30] were triggered by matrix model approach to $2D$ quantum gravity [31]. Numerical studies [32, 33] have been recently performed, instead, to capture a double scaling limit in $2D$ TEK models, motivated by noncommutative quantum field theory. In particular Wilson loops were computed in [33]: those results exhibit significant deviations from the usual behaviors.

In this paper we try to perform an analytical approach to the same problem: in particular our computations should be considered as a first step in trying to recover the results presented in [17] and [23, 27] starting from TEK models. We point out here a basic difference between the compact and non-compact case: to recover the partition function on the noncommutative torus, one should have in fact to resort to the *constrained* TEK model proposed in [11]. There a peculiar double scaling limit was claimed to reproduce the theory at finite area, the dimension of the matrices and the structure of the twists being determined by Diophantine equations. The theory on the noncommutative plane is instead reached by a different double scaling limit and without resorting to any constraint. The compact case is definitely subtler: in particular it should account for non-trivial topological charges. The general description of the two-dimensional case requires, in fact, the presence of two different integers, usually denoted as (p, q), classifying the inequivalent projective modules on a fixed noncommutative torus [26, 34]. In order to perform concrete computations, an explicit parametrization of constrained TEK variables and an efficient procedure to implement the double scaling limit are welcome. We have therefore decided to reconsider the TEK formulation of gauge theories on the noncommutative two-dimensional torus, starting from a slightly different point of view with respect to the approach of [11]. The case of the noncommutative plane is easily recovered within the same framework. After, we have focused our attention on the structure of the classical solutions of the model and on their behavior under double scaling limits. The main result is that we are able to reproduce, within this framework, the whole tower of instanton solutions on the (p, q) projective module [26], on which the partition function has been shown to be localized [23]. The fluxon configurations emerging on the noncommutative plane [17] are as obtained as well, changing accordingly the double scaling limit.

The plan of the paper is the following: In section 2 we start by considering the structure of toroidal noncommutative lattices and using reducible representations of Weyl-'t Hooft algebra we are able to construct derivations endowed with an arbitrary constant curvature. They naturally depends on a pair of integers that will be interpreted as the topological charges classifying the projective modules in the continuum limit. We define the gauge theory through a one-plaquette Wilson action, obtaining a related family of TEK models: the dimension of the matrices and the structure of the twists nicely encode the topological content of the discretized modules. The equivalence of our unconstrained systems with the constrained ones proposed in reference [11] is then carefully discussed. In section 3 we analyze the continuum limits and we derive explicitly the relevant scalings. Section 4 is devoted to the solutions of the classical equations of motion of the two-dimensional twisted Eguchi-Kawai model, showing that they fall into two different families, distinguished by their matrix structure. In section 5 we describe their double scaling behaviors: only one family is important here, the other having infinite action in both limits. We find all the solutions having finite action and we discuss their equivalence with the instanton solutions on the noncommutative torus and with the fluxons on the noncommutative plane. The possibility to compute the partition function directly from the TEK formulation is discussed in section 6: we remark the differences with the (exactly solvable) commutative case and we outline a possible strategy. As a consistency check of the TEK approach, we perform the computation, by matrix model technique, of the quantum average of a Wilson loop of vanishing area but infinitely winding. In this limit the calculation can be done exactly and we recover the known result of the noncommutative plane. In section 7 we draw our conclusions and we present the possible extensions of this work. In appendix A we show that Morita equivalent theories are described, in our formalism, by the very same matrix model. Appendix B is instead devoted to the explicit construction of the Wilson lattice action associated to our TEK models by means of a discretized version of the familiar star-product.

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2. Noncommutative gauge theories in two dimensions and TEK models

Large-N reduced models combine some powerful approaches to quantum field theories at non-perturbative level: the $1/N$-expansion, the lattice approximation to space-time, the loop equations and the matrix model techniques. The basic idea, dating back to Eguchi and Kawai [35], is that standard U(N) and SU(N) lattice gauge theories may be equivalent to their reduction to one plaquette in the large-N limit. Translations, encoding space-time dependence, naturally emerge as particular transformations inside the huge internal symmetry group. The possibility that large matrices could dynamically generate the space-time has been also vigorously advocated in [36, 37]. In its original formulation the plaquette action for Wilson lattice theory simplifies to

$$S_{EK}(U_\mu) = -N\beta \sum_{\mu\neq\nu} \mathrm{Tr}\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right), \tag{2.1}$$

where $\mu, \nu = 1, \ldots, D$ and U_μ being U(N) matrices. If the $\mathrm{U}(1)^D$ symmetry of the action (2.1) is not spontaneously broken, Schwinger-Dyson equations are unaltered under reduction process: unluckily this is not always true in $D > 2$, the equivalence holding only in the strong-coupling regime [38]. The two-dimensional case has been studied in the eighties [39] and more recently in [40], leading to contradictory conclusions.

In order to avoid the problem with the limited viability of the model defined in eq. (2.1) it was proposed, in even dimensions, the TEK model [12]

$$S_{TEK}(U_\mu) = -N\beta \sum_{\mu\neq\nu} \mathcal{Z}_{\mu\nu}\, \mathrm{Tr}\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right), \tag{2.2}$$

where the factor $\mathcal{Z}_{\mu\nu}$ is the *twist*,

$$\mathcal{Z}_{\mu\nu} = \mathcal{Z}^*_{\nu\mu} = \exp\left(\frac{2\pi i k_{\mu\nu}}{N}\right). \tag{2.3}$$

Here $k_{\mu\nu}$ is an integer-valued antisymmetric matrix and U_μ is usually restricted to SU(N). The $\mathrm{U}(1)^D$ symmetry is now reduced to $(Z_N)^D$ symmetry: it can be shown (see for example [12, 13]) that Schwinger-Dyson equations still hold in the weak-coupling regime due to the non-trivial vacuum structure. Numerical studies confirmed the equivalence with Wilson lattice gauge theory but the initial hope for a deeper understanding of confinement properties were disappointed and the interest for the model faded away.

The situation changed recently thanks to a new interpretation of the TEK model as a non-perturbative description of noncommutative gauge theory given by the action (we consider here the simplest case, namely U(1))

$$S = \frac{1}{4g^2}\int dx^D F_{\mu\nu}(x)\star F^{\mu\nu}(x)\,, \tag{2.4}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu \star A_\nu - A_\nu \star A_\mu)\,. \tag{2.5}$$

The star-product is defined as

$$f(x) * g(x) = \exp\left(i\frac{\theta^{\mu\nu}}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)|_{y=x}\,, \tag{2.6}$$

the noncommutative parameter $\theta^{\mu\nu}$ being an antisymmetric matrix with dimensions of a length square. Noncommutative gauge invariance is realized through star-unitary transformations $\mathrm{U}(x) \star \mathrm{U}(x)^\dagger = \mathrm{U}(x)^\dagger \star \mathrm{U}(x) = 1$

$$A'_\mu(x) = \mathrm{U}(x) \star A_\mu(x) \star \mathrm{U}(x)^\dagger + i\,\mathrm{U}(x) \star \partial_\mu\,\mathrm{U}(x)^\dagger\,. \tag{2.7}$$

In refs. [11] it was shown that the lattice version of the action eq. (2.4) turns out to be equivalent to certain reduced twisted $\mathrm{U}(N)$ models at finite N: the formulation has been given on a periodic lattice and the noncommutative parameter $\theta^{\mu\nu}$ is forced to take discrete values. The continuum limit of lattice noncommutative gauge theory coincides precisely with the large-N limit of the reduced (twisted) model. The $\mathrm{SU}(N)$ symmetry of the TEK action corresponds to the gauge invariance in the noncommutative gauge theories, realized through the star-unitary transformations. The novelty, observed in refs. [11], is that one has to take the large-N limit in a very peculiar way to land on noncommutative ground, starting from specific finite-N TEK formulations. We are going to discuss in details the two-dimensional case, that is our main concern here. As we have anticipated in the introduction, we start from the very beginning, trying to exploit as much as possible the algebraic properties of the "fuzzy" torus. The final results of our construction are equivalent to the ones presented in [11], but we feel that our approach is in some sense complementary and more suitable to perform our computations.

2.1 Gauge theories on a noncommutative toroidal lattice: the unconstrained formulation

We start by recalling some elementary definitions and by introducing the basic formalism that we will extend in the noncommutative case: a *commutative* toroidal square lattice is a set of vectors of the form

$$\vec{x} = a\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \equiv a\vec{n}\,, \tag{2.8}$$

endowed with the equivalence relation $\vec{x} \sim \vec{y}$ iff $\vec{x} - \vec{y} = Na\vec{l}$. Here $\vec{n}$ and $\vec{l}$ are integer-valued vectors, while the parameter a is identified with the lattice spacing. A basis for the functions defined on this lattice is given by

$$u_{\vec{k}}(\vec{x}) = \exp(i\vec{k}\cdot\vec{x})\,, \tag{2.9}$$

where the vector $\vec{k}$ belongs to the dual lattice

$$\vec{k} = \frac{2\pi\vec{m}}{Na}\,, \tag{2.10}$$

$\vec{m}$ being a vector of integer numbers. The period of the momentum lattice is $2\pi/a$, namely

$$u_{\vec{k}+2\pi/a}(\vec{x}) = u_{\vec{k}}(\vec{x})\,. \tag{2.11}$$

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The completeness of this basis is expressed by the condition

$$\frac{1}{N^2}\sum_{\vec{k}} u^*_{\vec{k}}(\vec{x})u_{\vec{k}}(\vec{y}) = \delta^P_{\vec{x},\vec{y}}\,, \tag{2.12}$$

while the orthogonality relation is given by

$$\frac{1}{N^2}\sum_{\vec{x}} u^*_{\vec{k}'}(\vec{x})u_{\vec{k}}(\vec{x}) = \delta^P_{\vec{k},\vec{k}'}\,. \tag{2.13}$$

Here the symbols $\delta^P_{\vec{r},\vec{s}}$ denotes the periodic Kronecker delta: this function is equal to one when $\vec{r}$ and $\vec{s}$ differ by any integer multiple of the period of the lattice that they span and zero otherwise. Any function $f(\vec{x})$ defined on this lattice can be Fourier-expanded on this basis

$$f(\vec{x}) = \frac{1}{N}\sum_{\vec{k}} f_{\vec{k}}\, u_{\vec{k}}(\vec{x})\,, \tag{2.14}$$

where

$$f_{\vec{k}} = \frac{1}{N}\sum_{\vec{x}} f(\vec{x})\, u_{\vec{k}}(\vec{x})\,. \tag{2.15}$$

The algebra of functions over a toroidal bidimensional lattice is completely defined by the following property of this basis:

$$u_{\vec{k}}(\vec{x})u_{\vec{k}'}(\vec{x}) = u_{\vec{k}+\vec{k}'}(\vec{x}) = u_{\vec{k}'}(\vec{x})u_{\vec{k}}(\vec{x})\,. \tag{2.16}$$

In fact, by means of eq. (2.16), the Fourier coefficients of the product, $h(x)$, of two functions $f(x)$ and $g(x)$ are

$$h_{\vec{k}} = \frac{1}{N}\sum_{\vec{p}+\vec{q}=\vec{k}} f_{\vec{p}}\, g_{\vec{q}}\,, \tag{2.17}$$

i.e. the convolution of the original coefficients.

Each element in the present basis is also an eigenstate of the translation operators T_1 and T_2,

$$T^\dagger_{1,2} u_{\vec{k}}(\vec{x}) T_{1,2} = u_{\vec{k}}(\vec{x} + a\vec{\delta}_{1,2}) = e^{i\vec{k}\cdot\vec{\delta}_{1,2}} u_{\vec{k}}(\vec{x})\,, \tag{2.18}$$

where $\vec{\delta}_1 \equiv a(1,0)$ and $\vec{\delta}_2 \equiv a(0,1)$.

Although the definition eq. (2.8) is the most simple for a commutative lattice, it is not the natural one in the noncommutative langauge where algebraic relations play a more fundamental role. In this spirit, one can characterize a commutative square lattice through the abstract algebra eq. (2.16) in a fixed point

$$u_{\vec{k}} u_{\vec{k}'} = u_{\vec{k}+\vec{k}'} = u_{\vec{k}'} u_{\vec{k}} \tag{2.19}$$

and then one can reconstruct the basis in the other sites by means of the condition eq. (2.18) written in the form

$$T^\dagger_1 u_{\vec{k}} T_1 = e^{i\vec{k}\cdot\vec{\delta}_1} u_{\vec{k}} \qquad \text{and} \qquad T^\dagger_2 u_{\vec{k}} T_2 = e^{i\vec{k}\cdot\vec{\delta}_2} u_{\vec{k}}\,. \tag{2.20}$$

The usual geometrical representation eq. (2.8) will appear when we try to realize explicitly eq. (2.19) and eq. (2.20). We shall call *noncommutative* square lattice any representation of the deformed algebra

$$\mathcal{U}_{\vec{k}}\mathcal{U}_{\vec{k}'} = \exp\left(\pi i\Theta(k_1k_2' - k_2k_1')\right)\mathcal{U}_{\vec{k}+\vec{k}'} = \exp\left(2\pi i\Theta(k_1k_2' - k_2k_1')\right)\mathcal{U}_{\vec{k}'}\mathcal{U}_{\vec{k}} \tag{2.21}$$

and of the relations

$$T_1^\dagger\mathcal{U}_{\vec{k}}T_1 = e^{i\vec{k}\cdot\vec{\delta}_1}\mathcal{U}_{\vec{k}} \qquad \text{and} \qquad T_2^\dagger\mathcal{U}_{\vec{k}}T_2 = e^{i\vec{k}\cdot\vec{\delta}_2}\mathcal{U}_{\vec{k}}\,. \tag{2.22}$$

Obviously, this is a sensible definition only if it is not affected by the periodicity of the momenta $\vec{k} \to \vec{k} + 2\pi/a$. This imposes that

$$\frac{2\pi^2\Theta}{Na^2} = r \qquad \text{(an integer)} \qquad \Rightarrow \Theta = \frac{Nr}{2\pi^2}a^2\,. \tag{2.23}$$

In order to study the representations of the algebra (2.21), it is useful to rewrite everything in terms of adimensional quantities, namely introducing explicitly the integer vectors $\vec{m}$ defined in eq. (2.10). Now the algebra eq. (2.21) and the relations eq. (2.22) reads

$$\begin{aligned} \mathcal{U}_{\vec{m}}\,\mathcal{U}_{\vec{m}'} &= e^{\pi i\theta(m_1m_2'-m_2m_1')}\mathcal{U}_{\vec{m}+\vec{m}'} = e^{2\pi i\theta(m_1m_2'-m_2m_1')}\mathcal{U}_{\vec{m}'}\,\mathcal{U}_{\vec{m}}\,, \\ T_1^\dagger\mathcal{U}_{\vec{m}}T_1 &= e^{2\pi i m_1/N}\mathcal{U}_{\vec{m}} \qquad \text{and} \qquad T_2^\dagger\mathcal{U}_{\vec{m}}T_2 = e^{2\pi i m_2/N}\mathcal{U}_{\vec{m}}\,, \end{aligned} \tag{2.24}$$

with

$$\theta = \frac{2r}{N}\,. \tag{2.25}$$

The representations of eqs. (2.24) can be built starting from the two generators

$$\mathbb{U}_1 = \mathcal{U}_{(1,0)} \qquad \text{and} \qquad \mathbb{U}_2 = \mathcal{U}_{(0,1)}\,, \tag{2.26}$$

which obey the Weyl-'t Hooft algebra [41]

$$\mathbb{U}_1\,\mathbb{U}_2 = \exp\left(\frac{2\pi i 2r}{N}\right)\mathbb{U}_2\,\mathbb{U}_1 \tag{2.27}$$

as well as the constraints

$$T_1^\dagger\mathbb{U}_1T_1 = e^{2\pi i/N}\mathbb{U}_1 \qquad T_1^\dagger\mathbb{U}_2T_1 = \mathbb{U}_2 \qquad T_2^\dagger\mathbb{U}_1T_2 = \mathbb{U}_1 \qquad T_2^\dagger\mathbb{U}_2T_2 = e^{2\pi i/N}\mathbb{U}_2\,. \tag{2.28}$$

The generic operator $\mathcal{U}_{\vec{m}}$ is then easily realized as

$$\mathcal{U}_{\vec{m}} = \exp\left(-\frac{2\pi i r\, m_1m_2}{N}\right)\mathbb{U}_1^{m_1}\,\mathbb{U}_2^{m_2}\,. \tag{2.29}$$

We stress that we have reduced our original task, the construction of the noncommutative fuzzy torus, to a well-defined algebraic problem, namely to find the representation of the algebra eq. (2.27) and of the relations eqs. (2.28).

We first analyze the case of *irreducible* representations. It is well-known there exists only one irreducible representation of the algebra eq. (2.27): denoting with $l = \gcd(N, 2r)$, that can be characterized as follows ($2r' = 2r/l$)

$$\mathbb{U}_1 = (U_1^0)^{2r'} \qquad \text{and} \qquad \mathbb{U}_2 = U_2^0\,, \tag{2.30}$$

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where U_1^0 and U_2^0 are the fundamental twist-eaters, satisfying the basic relation ($N' = N/l$) $U_1^0 U_2^0 = \exp(2\pi i/N') U_2^0 U_1^0$. Its dimension is exactly N'. In the following, to avoid useless complications, we limit ourselves to the case where $2r$ and N are coprime, namely $l = 1$, $r' = r$, $N = N'$. Let us notice this means that our N is odd.

At this point we can also realize the translation operators T_i in terms of the fundamental twist-eaters. Let us define

$$T_1 = (U_2^0)^s \qquad \text{and} \qquad T_2 = (U_1^0)^\dagger \,, \tag{2.31}$$

where the integers s and k satisfies the Diophantine equation[1]

$$(2r)s - kN = 1 \,. \tag{2.32}$$

Then T_1 obviously commutes with $\mathbb{U}_2$ and

$$T_1^\dagger \mathbb{U}_1 T_1 = (U_2^0)^{s\dagger} (U_1^0)^{2r} (U_2^0)^s = e^{2\pi i (2rs)/N} \mathbb{U}_1 = e^{2\pi i (kN+1)/N} \mathbb{U}_1 = e^{2\pi i/N} \mathbb{U}_1 \,. \tag{2.33}$$

In the same way it is manifest that T_2 commutes with $\mathbb{U}_1$, while

$$T_2^\dagger \mathbb{U}_2 T_2 = U_1^0 U_2^0 (U_1^0)^\dagger = e^{2\pi i/N} \mathbb{U}_2 \,. \tag{2.34}$$

It is intriguing to notice that the operators T_i define a noncommutative lattice whose parameter θ' is

$$T_1 T_2 = (U_2^0)^s (U_1^0)^\dagger = \exp\left(\frac{2\pi i s}{N}\right) (U_1^0)^\dagger (U_2^0)^s = \exp\left(2\pi i \left(\frac{1+kN}{2rN}\right)\right) T_2 T_1 \,. \tag{2.35}$$

The role of translation operators is instead played by $\mathbb{U}_1^\dagger$ and $\mathbb{U}_2^\dagger$ respectively. These two tori can be mapped one into the other through a particular transformation: let us define the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -kN + 4r^2 & N(s-2r) \\ k(2r-s) & s^2 - kN \end{pmatrix} \tag{2.36}$$

that belongs to $\mathrm{SL}(2,\mathbb{Z})$, namely

$$ad - bc = (4r^2 - kN)(s^2 - kN) + Nk(s-2r)^2 = (kN - 2rs)^2 = 1 \,. \tag{2.37}$$

This simple modular transformation maps the noncommutative parameter θ into

$$\theta' = \frac{c + d\theta}{a + b\theta} = \frac{cN + 2rd}{aN + 2rb} = \frac{s}{N} = \frac{1+kN}{2rN} \,, \tag{2.38}$$

while the volume of the two lattices is trivially the same. This is an example of the so-called Morita equivalence [18], which has played a relevant role in the recent approaches to NCYM_2 [16, 17, 23, 27]. How this symmetry emerges in the present framework is discussed in appendix A.

[1] A solution to this equation always exists since $2r$ and N are coprime.

The translations constructed above are the only ones possible if we strictly consider irreducible representations: the phase appearing in the commutation relation eq. (2.35) can be naturally associated to the (necessarily constant) curvature of the translation operators. Next we would like to construct translations with an arbitrary constant curvature, namely such that

$$T_1 T_2 = \exp\left(2\pi i \left(\frac{m}{n} + \frac{s}{N}\right)\right) T_2 T_1 = \exp\left(2\pi i \frac{(Nm+sn)}{(nN)}\right) T_2 T_1 \,, \tag{2.39}$$

where m and n are arbitrary integers, while s has been defined in eq. (2.32). This is naturally accomplished if we accept reducible representations of the twist-eaters algebra: in the general case the following relations have to be satisfied

$$\begin{aligned} \mathbb{U}_1 \mathbb{U}_2 &= e^{2\pi i (2r)/N} \mathbb{U}_2 \mathbb{U}_1 \,, \\ T_1 T_2 &= e^{2\pi i (Nm+sn)/(nN)} T_2 T_1 \,, \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} T_1^\dagger \mathbb{U}_1 T_1 &= e^{2\pi i/N} \mathbb{U}_1 \,, \qquad T_1^\dagger \mathbb{U}_2 T_1 = \mathbb{U}_2 \,, \\ T_2^\dagger \mathbb{U}_2 T_2 &= e^{2\pi i/N} \mathbb{U}_2 \,, \qquad T_2^\dagger \mathbb{U}_1 T_2 = \mathbb{U}_1 \,, \end{aligned} \tag{2.41}$$

which can be seen as a four dimensional noncommutative torus, whose Θ-matrix is not in a canonical form. To represent the above structure, we find useful to introduce the new translation operators $\tilde{T}_i$

$$\tilde{T}_1 = \mathbb{U}_2^{\dagger s} \, T_1 \,, \qquad \text{and} \qquad \tilde{T}_2 = \mathbb{U}_1 T_2 \,. \tag{2.42}$$

These operators satisfy a simpler algebra, namely

$$\begin{aligned} \tilde{T}_1^\dagger \mathbb{U}_1 \tilde{T}_1 &= T_1^\dagger \, \mathbb{U}_2^s \mathbb{U}_1 \mathbb{U}_2^{\dagger s} \, T_1 = e^{-2\pi i (2rs)/N} T_1^\dagger \mathbb{U}_1 T_1 = \mathbb{U}_1 \,, \\ \tilde{T}_2^\dagger \mathbb{U}_2 \tilde{T}_2 &= T_2^\dagger \, \mathbb{U}_1^\dagger \mathbb{U}_2 \mathbb{U}_1 \, T_2 = e^{-2\pi i/N} T_2^\dagger \mathbb{U}_2 T_2 = \mathbb{U}_2 \,, \end{aligned} \tag{2.43}$$

where the Diophantine equation has been used. Obviously we also have that

$$\tilde{T}_1^\dagger \mathbb{U}_2 \tilde{T}_1 = \mathbb{U}_2 \qquad \text{and} \qquad \tilde{T}_2^\dagger \mathbb{U}_1 \tilde{T}_2 = \mathbb{U}_1 \,. \tag{2.44}$$

All these equations can be summarized in the statement that the operators $\tilde{T}_i$ commute with the coordinates $\mathbb{U}_i$. The operators $\tilde{T}_i$ define an independent bidimensional torus, whose noncommutative parameter θ is determined by

$$\tilde{T}_1 \tilde{T}_2 = e^{2\pi i ((mN+sn)/(nN) - s/N)} \tilde{T}_2 \tilde{T}_1 = e^{2\pi i m/n} \tilde{T}_2 \tilde{T}_1 \,. \tag{2.45}$$

We finally end up with the announced reducible case: the description of the general relations eq. (2.40) and eq. (2.41) is encoded into two disjoint noncommutative tori, whose θ's are respectively

$$\theta = \frac{2r}{N} \qquad \text{and} \qquad \theta_T = \frac{m}{n} \,. \tag{2.46}$$

The cheapest way to obtain a representation of the complete algebra is, in fact, to take the tensor product of the representations of the two tori. The dimension of a generic representation is a multiple of $(n'N)$ where $n' = n/\gcd(n,m)$.

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A better understanding of the geometrical meaning of this construction can be obtained using the following parametrization for the integers m and n,

$$n = Np - (2r)q \qquad m = -sp + kq\,, \tag{2.47}$$

where p and q are arbitrary integers. This parametrization is absolutely general being inverted by

$$p = -2rm - kn \qquad q = -mN - ns\,. \tag{2.48}$$

These two equations also imply the interesting property that the $\gcd(m,n) = \gcd(p,q)$. The geometrical data $\mathcal{N} = \gcd(m,n) = \gcd(p,q)$ can be identified with the rank of the gauge group of the noncommutative theory [23].

In our approach n is a multiple of the dimension of the representation of the second torus, therefore it is a positive number; in terms of p and q this condition reads

$$n = Np - (2r)q = N(p - \theta q) > 0\,. \tag{2.49}$$

This condition, in the continuum, will appear as the positive cone constraint and will determine (p,q) associated to inequivalent projective modules. The value of the θ of the original T_i operators is instead

$$\frac{Nm + sn}{nN} = -\frac{q}{N^2(p - \theta q)}\,. \tag{2.50}$$

We have now to exhibit an explicit representation of eq. (2.40): the natural choice is given by

$$\mathbb{U}_1 = (U_1^0)^{2r} \otimes \mathbb{I}_{n'}\,, \qquad \mathbb{U}_2 = (U_2^0) \otimes \mathbb{I}_{n'}\,, \tag{2.51}$$

$$T_1 = (U_2^0)^s \otimes \Gamma_1^{m'}\,, \qquad T_2 = (U_1^0)^\dagger \otimes \Gamma_2\,, \tag{2.52}$$

where $\Gamma_1\Gamma_2 = \exp(2\pi i/n')\Gamma_2\Gamma_1$ and $m' = m/\gcd(n,m)$.

In the following we shall interpret the two integers p and q introduced above as the same p and q that classify the modules on the noncommutative torus and we shall construct the gauge theories on these modules. Let us start with the simple trivial module, namely that with vanishing q and $p = 1$:

(1,0): we choose $n = n' = N$ and $m = m' = -s$. In this module the fundamental translation operators commutes and their form is simply

$$T_1 = (U_2^0)^s \otimes (U_2^0)^{-s} \qquad T_2 = (U_1^0)^\dagger \otimes (U_1^0)^\dagger\,. \tag{2.53}$$

Then the coordinates $\mathbb{U}_i$ are

$$\mathbb{U}_1 = (U_1^0)^{2r} \otimes \mathbb{I}_N\,, \qquad \mathbb{U}_2 = (U_2^0) \otimes \mathbb{I}_N\,. \tag{2.54}$$

To define a gauge theory we must construct the most general translation operators, which satisfy the algebra eq. (2.40) and eq. (2.41) except for the constant curvature condition. We call these operators D_i and they are simply given by

$$D_1 = (U_2^0)^s \otimes V_1 \qquad D_2 = (U_1^0)^\dagger \otimes V_2\,, \tag{2.55}$$

where V_1 and V_2 are arbitrary $N \times N$ unitary matrices, encoding the degrees of freedom of the related gauge theory.

The next step, in our construction, is to write down an action: the natural one is the generalized Wilson one-plaquette action, whose form is

$$S = \alpha - \beta\Big(\exp(2i\phi)\,\mathrm{Tr}[D_1 D_2 D_1^\dagger D_2^\dagger] + \exp(-2i\phi)\,\mathrm{Tr}[D_2 D_1 D_2^\dagger D_1^\dagger]\Big)\,. \tag{2.56}$$

The parameters α and β are real numbers, while ϕ is a background phase.[2] The constant α is fixed by requiring that S be positive definite. The action can be written in the following equivalent manner

$$S = \beta\,\mathrm{Tr}\Big[\Big(\exp(i\phi)D_1D_2 - \exp(-i\phi)D_2D_1\Big)\Big(\exp(-i\phi)D_2^\dagger D_1^\dagger - \exp(i\phi)D_1^\dagger D_2^\dagger\Big)\Big] + \alpha - 2\beta N^2\,, \tag{2.57}$$

suggesting the simple choice

$$\alpha = 2\beta N^2\,, \tag{2.58}$$

combined with the requirement that $\beta > 0$. We arrive therefore to the nice expression

$$S = \beta\,\mathrm{Tr}\Big[\Big(\exp(i\phi)D_1D_2 - \exp(-i\phi)D_2D_1\Big)\Big(\exp(-i\phi)D_2^\dagger D_1^\dagger - \exp(i\phi)D_1^\dagger D_2^\dagger\Big)\Big]\,, \tag{2.59}$$

where ϕ is still undetermined: we choose the background phase by requiring that the absolute minimum, i.e. the one with zero action, corresponds to the derivatives with constant curvature on the module, in complete analogy with the continuum description. In our case, due to the fact we have commuting derivatives, $\phi = 0$ modulo N. The final action turns out to be

$$\begin{aligned} S &= \beta\,\mathrm{Tr}\Big[\Big(D_1D_2 - D_2D_1\Big)\Big(D_2^\dagger D_1^\dagger - D_1^\dagger D_2^\dagger\Big)\Big] \\ &= \beta N\,\mathrm{Tr}\Big[\Big(\exp\Big(\frac{\pi i s}{N}\Big)V_1V_2 - \exp\Big(-\frac{\pi i s}{N}\Big)V_2V_1\Big)\times \\ &\qquad \Big(\exp\Big(-\frac{\pi i s}{N}\Big)V_2^\dagger V_1^\dagger - \exp\Big(\frac{\pi i s}{N}\Big)V_1^\dagger V_2^\dagger\Big)\Big]\,. \end{aligned} \tag{2.60}$$

In terms of the V_i matrices the theory is the well-known TEK model, described before, with a twist factor given by $\exp(2\pi i s/N)$. It is important to notice the appearing of the factor N in front to the classical action: it carries part of the space-time dependence, being in fact factored out as the volume of the translations.

We are ready now to consider the general case.

(p,q): For simplicity, we shall take p and q to be coprime or equivalently the gauge group to be U(1), then $n = Np - 2rq$ and $m = kq - ps$ are coprime. In this case the fundamental translation operators are represented as

$$T_1 = (U_2^0)^s \otimes \Gamma_1^{kq-ps} \qquad T_2 = (U_1^0)^\dagger \otimes \Gamma_2\,, \tag{2.61}$$

where the dimension of Γ_i is $n = Np - 2rq$ since $\gcd(m,n) = \gcd(p,q) = 1$. The most general derivatives on this module can be written as

$$D_1 = (U_2^0)^s \otimes V_1 \qquad D_2 = (U_1^0)^\dagger \otimes V_2\,, \tag{2.62}$$

[2]The introduction of a background phased is suggested by the possible presence of a background flux in the continuum limit.

where now the matrices V_i are $(Np - (2rq)) \times (Np - (2rq))$ unitary matrices. The corresponding action is again

$$S = \beta \,\mathrm{Tr}\left[\left(\exp(i\phi) D_1 D_2 - \exp(-i\phi) D_2 D_1\right)\left(\exp(-i\phi) D_2^\dagger D_1^\dagger - \exp(i\phi) D_1^\dagger D_2^\dagger\right)\right], \quad (2.63)$$

where ϕ is now different from zero modulo N and determined by the condition that the absolute minimum is the derivation corresponding to the constant curvature derivative. Its value is

$$\phi = -\pi\left(\frac{m}{n} + \frac{s}{N}\right) \qquad \text{modulo } N\,. \quad (2.64)$$

In complete analogy with the trivial case, we can perform part of the traces, writing the action in terms of the fundamental variables V_1, V_2 carrying the dynamical degrees of freedom. In doing so β is again renormalized by a factor N, a crucial feature when discussing the continuum limit. We end up with a TEK action twisted by the phase

$$\phi' = -\pi\frac{m}{n} = -\pi\frac{kq - sp}{Np - 2rq}\,, \quad (2.65)$$

while the dimension of the matrices is equal to $Np - 2rq$, i.e.

$$S = \beta N \,\mathrm{Tr}\left[\left(\exp(i\phi') V_1 V_2 - \exp(-i\phi') V_2 V_1\right)\left(\exp(-i\phi') V_2^\dagger V_1^\dagger - \exp(i\phi') V_1^\dagger V_2^\dagger\right)\right]. \quad (2.66)$$

This is, of course, the final result we were looking for and it deserves some comments. First of all we remark that we were able to write the discretized theory in terms of a "conventional" TEK model, in the sense no constraint is required on the matrices V_i. Once we have fixed the number of lattice points, N^2, and the noncommutative (adimensional) parameter, $\theta = 2r/N$, all the (p, q) modules satisfying the positivity constraint eq. (2.49) are simply described. The topological distinction is effectively encoded on the size of the matrices and on the twist-phase characterizing the matrix model action. The second point we would like to stress is concerning our actual choice of the background phase: we have explicitly tuned the absolute minimum of the classical action with the constant curvature connection of the associated (discretized) projective module, a well-known property of the continuum description [26]. We alert the reader that other possibilities are in principle allowed, as we will see, consistently with recovering this crucial property in the continuum limit. Finally we stress the explicit appearance of the factor N in front of the action, due to a partial decoupling of the space-time structure in taking the traces: it will play a relevant role in constructing the continuum limit.

2.2 The equivalence with the constrained formulation

In the literature, another TEK model describing the Yang-Mills theory in two dimensions over the generic module (p, q), at discretized level, has been discussed in ref. [11]. The model proposed there differs, apparently, in a fundamental aspect from the one we derived here: it is a *constrained* TEK model, *i.e.* the matrices entering in the classical action must satisfy a non-trivial equation. Although the two models are required to be relatively consistent only in the continuum limit, it is nevertheless important to find their precise relation directly at discretized level. In the following we will discuss explicitly the case $\theta \neq 0$.

In order to facilitate the comparison between our action and theirs, we shall begin by translating their model in our notation. Their starting point is a noncommutative lattice of dimension $N = \tilde{n}q$, generated by the matrices satisfying the Weyl-'t Hooft algebra

$$U_1U_2 = e^{2\pi i\theta}U_2U_1\,, \tag{2.67}$$

with[3]

$$\theta = \frac{p}{q} - \frac{\tilde{m}}{\tilde{n}q} = \frac{p\tilde{n} - \tilde{m}}{\tilde{n}q}\,. \tag{2.68}$$

The matrix action, defining their final model, is written as

$$S_1 = -\beta Z\,\mathrm{Tr}(D_1D_2D_1^\dagger D_2^\dagger) - \beta Z^*\,\mathrm{Tr}(D_2D_1D_2^\dagger D_1^\dagger)\,, \tag{2.69}$$

where the matrices D_i are subjected to the constraints

$$\begin{aligned} D_1^\dagger U_1 D_1 &= e^{2\pi i/(\tilde{n}q)}U_1\,, \qquad & D_2^\dagger U_1 D_2 &= U_1\,,\\ D_2^\dagger U_2 D_2 &= e^{2\pi i/(\tilde{n}q)}U_2\,, \qquad & D_1^\dagger U_2 D_1 &= U_2\,. \end{aligned} \tag{2.70}$$

The matrices D_i are therefore generators of translations. The parameter β is an overall normalization, while Z is a phase chosen, at the end of the day, to select the desired vacuum: we see the actions eq. (2.63) and eq. (2.69) are formally the same (a suitable subtraction can be easily performed in eq. (2.69) to make the action positive defined). The difference arises when it comes to representing explicitly the algebra eq. (2.67) and the matrices D_i satisfying the constraints eq. (2.70).

The algebra eq. (2.67) has been realized, in ref. [11], by means of matrices $(\tilde{m}\tilde{n}q^2) \times (\tilde{m}\tilde{n}q^2)$ given by

$$U_1 = (\Gamma_2)^{\tilde{m}} \otimes (\tilde{\Gamma}_1^\dagger)^p\,, \qquad U_2 = (\Gamma_1)^{\tilde{m}} \otimes (\tilde{\Gamma}_2^\dagger) \tag{2.71}$$

with

$$\Gamma_1\Gamma_2 = e^{2\pi i/(\tilde{m}\tilde{n}q)}\Gamma_2\Gamma_1 \qquad \tilde{\Gamma}_1\tilde{\Gamma}_2 = e^{2\pi i/q}\tilde{\Gamma}_2\tilde{\Gamma}_1\,, \tag{2.72}$$

and $(U_i)^{\tilde{n}q} = \mathbb{I}$. The matrices Γ_i and $\tilde{\Gamma}_i$ are the irreducible representations of the algebras eq. (2.72) and thus of dimensions $(\tilde{m}\tilde{n}q) \times (\tilde{m}\tilde{n}q)$ and $q \times q$ respectively. The solutions of eq. (2.70) are then parameterized in terms of a particular $\bar{D}_i$,

$$\bar{D}_1 = (\Gamma_1)^\dagger \otimes \mathbb{I}_q \qquad \bar{D}_2 = \Gamma_2 \otimes \mathbb{I}_q\,, \tag{2.73}$$

solving the equations, and the matrices that commute with U_i. In other words, the translation operators D_i are written as

$$D_i = S_i\bar{D}_i \qquad \text{where } S_iU_j = U_jS_i \qquad \forall\, i,j = 1,2\,. \tag{2.74}$$

The set of the matrices commuting with the generators U_i of the lattice are then parameterized with the help of two other generators Z_i,

$$Z_1 = (\Gamma_2)^{\tilde{n}} \otimes (\tilde{\Gamma}_1)^\dagger \qquad Z_2 = (\Gamma_1)^{\dagger\tilde{n}} \otimes (\tilde{\Gamma}_2)^{\hat{a}}\,, \tag{2.75}$$

[3] In the following, we shall assume $\gcd(p\tilde{n} - \tilde{m}, \tilde{n}q) = 1$.

with a defined by the Diophantine equation $\hat{a}p + bq = 1$. It is not difficult to verify that $Z_i U_j = U_j Z_i$ and

$$Z_1 Z_2 = e^{2\pi i\theta'} Z_2 Z_1 \,, \qquad \text{with } \theta' = \frac{\hat{a}\theta + b}{p - q\theta} \,. \tag{2.76}$$

We have also $(Z_i)^{\tilde{m}q} = \mathbb{I}$. The most general S_i is therefore a unitary matrix belonging to the algebra generated by eq. (2.76). Finally, one completes the definition of model by formally expressing the operators D_i in terms of unconstrained variables Z_i through the relation eq. (2.74). The procedure, however, cannot be carried out explicitly without loosing the elegant and simple structure of the matrix model. This feature makes not straightforward to use this model for concrete computations on the torus. Nevertheless, this TEK representation admits an elegant translation as the Wilson action living on the dual noncommutative lattices of the derivation endowed with the (discretized) star-product defined by θ'.

Our procedure would have, instead, produced a smaller unconstrained TEK model, with matrices V_i of dimension $n \times n$,

$$n = Np - 2rq = \tilde{n}qp - (p\tilde{n} - \tilde{m})q = \tilde{m}q \,, \tag{2.77}$$

and a twist-phase

$$\phi = \frac{m}{n} = -\frac{s}{\tilde{n}q} - \frac{1}{\tilde{m}\tilde{n}q} \,. \tag{2.78}$$

We have used here our Diophantine equation, that in terms of the new variables is

$$(p\tilde{n} - \tilde{m})s - k\tilde{n}q = \tilde{n}(ps - kq) - \tilde{m}s = -\tilde{n}m - \tilde{m}s = 1 \,. \tag{2.79}$$

Then its action is

$$S_2 = \beta\tilde{n}q \,\mathrm{Tr}\Big[\Big(\exp(\pi i\phi)V_1V_2 - \exp(-\pi i\phi)V_2V_1\Big)\Big(\exp(-\pi i\phi)V_2^\dagger V_1^\dagger - \exp(\pi i\phi)V_1^\dagger V_2^\dagger\Big)\Big] \,. \tag{2.80}$$

We shall try to answer the question on how these two models are related: in particular we shall explain that there is a change of variables that reduces the action eq. (2.69) to the action eq. (2.80).

The key point to notice is the following: we know, from the general theory, that there exists only one irreducible representation of the Weyl-'t Hooft algebra eq. (2.67) and that its dimension is $\tilde{n}q$. The unitary representation eq. (2.71) is then completely reducible and it can be written in the form

$$\begin{aligned} U_1 &= (\Gamma_2)^{\tilde{m}} \otimes (\tilde{\Gamma}_1^\dagger)^p = \Omega^\dagger(\bar{U}_1^{p\tilde{n}-\tilde{m}} \otimes \mathbf{d}^1_{\tilde{m}q})\Omega \\ U_2 &= (\Gamma_1)^{\tilde{m}} \otimes (\tilde{\Gamma}_2^\dagger) = \Omega^\dagger(\bar{U}_2 \otimes \mathbf{d}^2_{\tilde{m}q})\Omega \,, \end{aligned} \tag{2.81}$$

where $\bar{U}_i$ are the irreducible representation of the algebra $\bar{U}_1\bar{U}_2 = e^{2\pi i/(\tilde{n}q)}\bar{U}_2\bar{U}_1$. Moreover $\mathbf{d}^i_{\tilde{m}q}$ are two unitary diagonal matrices of dimension $\tilde{m}q \times \tilde{m}q$ and Ω is a unitary transformations: we shall show that each eigenvalue of the matrices U_i is degenerate $\tilde{m}q$ and therefore the matrices $\mathbf{d}^i_{\tilde{m}q}$ are actually proportional to the identity $\mathbb{I}_{\tilde{m}q}$.

To begin with, we consider the generator U_1 and the matrix Z_1 commuting with it. Let e_1 be one of the common eigenvectors,

$$U_1 e_1 = e^{2\pi i\alpha} e_1 \qquad \text{and} \qquad Z_1 e_1 = e^{2\pi i\beta} e_1 \,,$$

then the vectors $f_i = Z_2^{i-1} e_1$ $(i = 1, \ldots, \tilde{m}q)$ are $\tilde{m}q$ independent vectors. In fact, the algebra eq. (2.76) implies that f_i are eigenvectors of Z_1 corresponding to different eigenvalues. On the other hand, since Z_2 commutes with U_1, the vectors f_i are also eigenvectors of U_1 all with the same eigenvalue $e^{2\pi i\alpha}$. Thus each eigenvalue of U_1 is degenerate at least $\tilde{m}q$. To conclude that each eigenvalue is exactly $\tilde{m}q$ degenerate, we recall that the matrix U_1 has dimension $(\tilde{m}\tilde{n}q^2) \times (\tilde{m}\tilde{n}q^2)$ and that it possesses at least $\tilde{n}q$ different eigenvalues because of the algebra eq. (2.67).[4] A similar reasoning allows us to reach the same result for U_2, obtaining therefore

$$\begin{aligned} U_1 &= (\Gamma_2)^{\tilde{m}} \otimes (\tilde{\Gamma}_1^\dagger)^p = e^{i\phi_1} \Omega^\dagger (\bar{U}_1^{p\tilde{n}-\tilde{m}} \otimes \mathbb{I}_{\tilde{m}q}) \Omega \,, \\ U_2 &= (\Gamma_1)^{\tilde{m}} \otimes (\tilde{\Gamma}_2^\dagger) = e^{i\phi_2} \Omega^\dagger (\bar{U}_2 \otimes \mathbb{I}_{\tilde{m}q}) \Omega \,. \end{aligned} \tag{2.82}$$

This, in turn, implies that the matrices S_i must be of the form

$$S_i = \Omega^\dagger (\mathbb{I}_{\tilde{n}q} \otimes \mathcal{V}_i) \Omega \,, \tag{2.83}$$

where $\mathcal{V}_i$ are unconstrained unitary matrices of dimension $(\tilde{m}q) \times (\tilde{m}q)$. The generic derivation can be then written

$$D_i = S_i \bar{D}_i = \Omega^\dagger (\mathbb{I}_{\tilde{n}q} \otimes \mathcal{V}_i) \Omega \bar{D}_i = \Omega^\dagger (\mathbb{I}_{\tilde{n}q} \otimes \mathcal{V}_i) \bar{\mathcal{D}}_i \Omega \,, \tag{2.84}$$

where $\bar{\mathcal{D}}_i = \Omega \bar{D}_i \Omega^\dagger$. The matrices $\bar{\mathcal{D}}_i$ are a particular solution of the constraints eq. (2.70) with

$$\bar{U}_1^{p\tilde{n}-\tilde{m}} \otimes \mathbb{I}_{\tilde{m}q} \equiv \mathcal{U}_1 \,, \qquad \bar{U}_2 \otimes \mathbb{I}_{\tilde{m}q} \equiv \mathcal{U}_2 \,. \tag{2.85}$$

The background translations $\bar{\mathcal{D}}_i$ possess the same factorized structure of the coordinates $\mathcal{U}_i$: this can be shown by writing $\bar{\mathcal{D}}_i$ as follows

$$\bar{\mathcal{D}}_1 = (\bar{U}_2^s \otimes \mathbb{I}_{\tilde{m}q}) \mathrm{R}_1 \qquad \bar{\mathcal{D}}_2 = (\bar{U}_1^\dagger \otimes \mathbb{I}_{\tilde{m}q}) \mathrm{R}_2 \,, \tag{2.86}$$

where the first term is an alternative solution of the constraint (2.70) for the $\mathcal{U}_i$. This, in turn, implies that the matrices $\mathrm{R_i}$ commutes with the $\mathcal{U}_i$ and thus they can be written as

$$\mathrm{R_i} = \mathbb{I}_{\tilde{n}q} \otimes \mathcal{W}_i \,. \tag{2.87}$$

[4]That two matrices U_i satisfying the Weyl-'t Hooft algebra,

$$U_1 U_2 = e^{2\pi i m/n} U_2 U_1 \qquad \text{with } \gcd(m,n) = 1 \,,$$

possess at least n distinct eigenvalues, is a trivial consequence of the following observation. Given an eigenvector e_1 of U_1, the vectors $(U_2)^{i-1} e_1$ $(i = 1, \ldots, n)$ are still eigenvectors of U_1 with different eigenvalues because of the Weyl-'t Hooft algebra. In particular, in the only irreducible representation, which has dimension n, all the eigenvalues are non-degenerate.

In other words, we have shown that the matrices $\bar{\mathcal{D}}_i$ are of the form

$$\bar{\mathcal{D}}_1 = \bar{U}_2^s \otimes \mathcal{W}_1 \qquad \bar{\mathcal{D}}_2 = \bar{U}_1^\dagger \otimes \mathcal{W}_2 \,, \tag{2.88}$$

with $\mathcal{W}_1\mathcal{W}_2 = e^{-2\pi i m/(\tilde{m}q)}\mathcal{W}_2\mathcal{W}_1$ since $\bar{D}_1\bar{D}_2 = e^{-2\pi i/(\tilde{m}\tilde{n}\tilde{q})}\bar{D}_2\bar{D}_1$. The general derivation can be then parameterized as follows (we obviously have defined $\mathcal{D}_i = \Omega D_i \Omega^\dagger$)

$$D_1 = \Omega^\dagger \mathcal{D}_1 \Omega = \Omega^\dagger(\bar{U}_2^s \otimes \mathcal{W}_1\mathcal{V}_1)\Omega \qquad D_2 = \Omega^\dagger \mathcal{D}_2 \Omega = \Omega^\dagger(\bar{U}_1^\dagger \otimes \mathcal{W}_2\mathcal{V}_2)\Omega \,, \tag{2.89}$$

Substituting this representation in the action eq. (2.69), it reduces to

$$S_1 = -\beta\,(\tilde{n}q) Z\; e^{2\pi i s/(\tilde{n}q)}\,\mathrm{Tr}\Big[\mathcal{W}_1\mathcal{V}_1\mathcal{W}_2\mathcal{V}_2\mathcal{V}_1^\dagger\mathcal{W}_1^\dagger\mathcal{V}_2^\dagger\mathcal{W}_2^\dagger\Big] + \text{c.c.}\,. \tag{2.90}$$

Changing variables to $V_i = \mathcal{V}_i\mathcal{W}_i$ and keeping in mind that Z was chosen in [11] as $Z = e^{-2\pi i/(\tilde{m}\tilde{n}q)}$, this action becomes identical to that in eq. (2.80) up to the background phase and a (trivial) subtraction term. One can check that the basic difference corresponds to a different choice of the configuration of minimal action.

Summarizing, the models are completely equivalent when θ is different from zero. They describe the same physics in different basis. In particular, the constrained basis of ref. [11] is the natural one for obtaining a (discretized) star-product interpretation of the model while ours is the most suitable for performing the matrix model computations, being in fact completely unconstrained.

3. The continuum limits

The next step of our construction is, of course, to recover the continuum limit: we will try to discuss the problem from a general point of view, having in mind the differences with the ordinary case and hoping to elucidate some subtle points. We start by recalling the basic relations between the size of our lattice, that we denote by L, and the dimensionful Θ parameter

$$\Theta = \frac{Nr}{2\pi^2}a^2 = \frac{2r}{N}\frac{L^2}{4\pi^2}\,, \tag{3.1}$$

$$L = Na\,. \tag{3.2}$$

Noncommutativity necessary implies that Θ has to be finite as N becomes large: we see that the situation drastically changes if we also require L to be finite (noncommutative torus) or not (noncommutative plane), reflecting into a non-trivial scaling of the parameter r. Finiteness of L, in fact, determines the continuum limit ($a \to 0$) as

$$a \simeq \frac{L}{N}\;: \tag{3.3}$$

the finiteness of Θ, therefore, implies that as $N \to \infty$

$$r \simeq \left(\frac{2\pi^2\Theta}{L^2}\right)N\,, \tag{3.4}$$

recovering the announced scaling of r. To reproduce the noncommutative plane we have, instead, to send L to infinity: this can be generically obtained by scaling for large-N

$$r \simeq N^{1-\gamma} \quad \Rightarrow \quad a \simeq \frac{1}{N^{1-\frac{\gamma}{2}}}, \tag{3.5}$$

$0 \leq \gamma \leq 1$. At this level, all these limits are potentially good in recovering the noncommutative plane. Usually people considered $\gamma = 1$ [11, 33]: it corresponds exactly to the limits explored in [15, 16, 17], where the size of the noncommutative torus has been assumed to scale as $\sqrt{N}$ in studying gauge theories. Actually, in the case of scalar theories in two dimensions, the possibility of more general scalings has been suggested in [42]: there, in order to cope with a solvable matrix model, the limit $\Theta \to \infty$ was also taken in a correlated way. The authors pointed out the possible appearance of three different phases, depending on the chosen scaling: more recently the scalar case was examined in [43] by means of a powerful matrix model technique, in the large-Θ limit too, and the existence of "exotic" scalings was also discussed. Related investigations in Yang-Mills theory have been performed in [44], where the possibility to consider more general scalings has been exploited in computing correlators of open Wilson lines. In the following we will restrict ourselves to the canonical case $\gamma = 1$, leaving the more general possibility to future investigations. Within this choice the partition function and Wilson lines correlators appear to be dominated by the classical solutions of the system [17], consistently with localization theorems [23].

The unexpected result of these analysis is that, in order to keep the noncommutative parameter finite as the lattice spacing goes to zero, we need forcing $N \to \infty$ simultaneously with $a \to 0$. At perturbative level this implies, generically, that non-planar diagrams no longer vanish: the usual proof of equivalence between TEK and Wilson lattice theory indeed requires that only planar diagrams survive the large-N limit and this is achieved by first taking N large and then going to continuum limit. We are searching here, instead, for a *double scaling* limit and, in particular, we are looking for non-perturbative effects studying the behavior of the classical solutions of the model.

A first interesting consequence is the following: in the non-compact case the space-time noncommutativity forces to correlate the ultraviolet ($a \to 0$) limit with the infrared ($L \to \infty$) limit: we have therefore a non-perturbative interpretation of the famous IR/UV connection, as pointed out in [11]. In the compact case the double scaling procedure has not a direct IR/UV interpretation: we have instead, in order to stay with θ finite, to perform a limit on the r parameter. Quite surprisingly this offers the possibility to recover, as a particular case, the commutative theory by simply tuning the limiting value of r. We will return on this opportunity in section 5, when discussing the classical solutions in the continuum limit.

Coming back to eq. (2.60) we see that, of course, the lattice spacing a does not appear explicitly in the definition of the matrix model: we have to identify the relation between β and a, establishing in this way the explicit double scaling limit to be performed in the TEK model. The following discussion extends the one presented in [33]. Dimensional considerations suggest a canonical scaling

$$\beta \sim \frac{1}{g^2 a^2}, \tag{3.6}$$

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g^2 being the physical coupling constant: we remark that in $D = 2$ the coupling of the noncommutative Yang-Mills theory has the dimension of a mass square. This choice deserves some comments. We recall, first of all, that the canonical scaling correctly reproduces the continuum limit of $2D$ *commutative* lattice gauge theories. TEK models are also seen to be equivalent to gauge theories on particular periodic lattices, written in the Wilson form through the discretized version of the noncommutative star-product [11]. It is quite natural, therefore, to assume eq. (3.6) in defining the continuum limit and we will adopt this point of view in the rest of the paper. This choice clearly determines the explicit form of the double scaling limit to be performed:

$$\begin{aligned} \beta &\simeq N^2 \qquad \text{NC torus}\,, \\ \beta &\simeq N \qquad \text{NC plane}\,. \end{aligned} \tag{3.7}$$

Having identified the precise form of the double scaling limit, we can try to get some intuitions on its physical consequences: we exploit the claimed equivalence of conventional TEK models with commutative large-N Yang-Mills theories. To this aim, let us review some well-known facts about the two dimensions.

Two-dimensional lattice gauge theories, at large-N, are known to have a non-trivial phase structure [45]: a third-order phase transition occurs at $\beta = 1/2$, distinguishing a strong-coupling regime ($\beta < 1/2$) from the (physical) weak-coupling regime ($\beta > 1/2$). The phase transition is reflected by a different functional form of the dimensionless string tension $k(\beta)$:

$$\begin{aligned} k(\beta) &= -\ln\beta \qquad\qquad \beta < \frac{1}{2} \\ k(\beta) &= -\ln\left(1 - \frac{1}{\beta}\right) \; \beta > \frac{1}{2}\,. \end{aligned} \tag{3.8}$$

The fact that the Wilson loops follows an exact area law, in the continuum limit, implies that one should tune the bare coupling constant β, as the continuum limit is approached ($a \to 0$), using the second relation in eq. (3.8)

$$a^2 = -\frac{1}{g^2}\ln\left(1 - \frac{1}{\beta}\right), \tag{3.9}$$

recovering the asymptotic behavior anticipated in eq. (3.6). From the point of view of ordinary gauge theories, the double scaling limit eq. (3.7) we have to consider goes deeply into the weak-coupling phase: therefore it is not expected to give new results in the Wilson theory, the extreme weak-coupling phase being dominated by the trivial classical solution $U = 1$ (in axial gauge). On the other hand, the situation should be rather different when our double scaling limit is applied to the TEK model: its weak-coupling regime has to be somehow different from conventional Wilson theory, at least at large β, if noncommutative theories have to be reproduced. We expect an highly non-trivial effect of eq. (3.7), augmented by the relevant scalings of the matrices and of the twist-phases, drastically changing the large-N behavior of the two-matrix model: in particular new non-trivial classical solutions could emerge from a saddle-point analysis. We remark that in [33] eq. (3.6) was derived from the scaling of the Wilson loop at small area: the large area behavior differs instead significantly from the commutative case. Before closing the section we have to mention that Wilson loops were also computed in perturbation theory [46], exhibiting an unexpected scaling as $\theta \to \infty$.

In order to confirm our expectations, we are going to study the classical solutions of the TEK model, in the double scaling limit relevant for the noncommutative theory: we assume that eq. (3.7) is valid and we will find the appearance of non-trivial configurations, potentially changing the structure of the weak-coupling regime of conventional two-dimensional lattice gauge theories.

4. Equations of motion and their solutions

In this section we solve the equations of motion for the TEK model in two dimensions: we can treat the problem in full generality, without referring to the particular dimension of the matrices or to the peculiar structure of the twists. The specific properties of the different models as well as the scaling of β in taking the continuum limit will be discussed in the next section. We start by considering the general action

$$S_{TEK} = \beta N \,\mathrm{Tr}\Big[\Big(e^{-\pi i m/n} V_1 V_2 - e^{\pi i m/n} V_2 V_1\Big)\Big(e^{i\pi m/n} V_2^\dagger V_1^\dagger - e^{-\pi i m/n} V_1^\dagger V_2^\dagger\Big)\Big], \qquad (4.1)$$

V_1, V_2 being unitary $n \times n$ matrices and m, n a couple of integers, that will be taken relatively prime. Eq. (4.1) makes also manifest the positive nature of S_{TEK}.

The model enjoys the gauge symmetry

$$V_1 \to C^\dagger V_1 C\,, \qquad \text{and} \qquad V_2 \to C^\dagger V_2 C\,, \qquad (4.2)$$

with C a unitary matrix: of course V_1 and V_2 do not transform under the U(1) subgroup, the effective symmetry group being therefore $\mathrm{SU}(n)/\mathbb{Z}_n$. The classical vacuum solution (namely the absolute minimum of the action) is determined by the condition

$$e^{-\pi i m/n} V_1^0 V_2^0 - e^{\pi i m/n} V_2^0 V_1^0 = 0\,, \qquad (4.3)$$

that can be solved in terms of twist-eaters (see eqs. (2.27) and (2.30)): it produces an irreducible representation of the two-dimensional Weyl-'t Hooft algebra, since $\gcd(m, n)$ is taken to be 1. The gauge inequivalent solutions of eq. (4.3) are labelled by the global U(1) phases multiplying V_1^0, V_2^0. We can therefore associate to the absolute minimum of the action eq. (4.1) a moduli space, with the topology of a torus, described by a pair of complex moduli $(z_1, z_2) \in \tilde{T}^2$.

The equations of motion defining all the other extrema are easily obtained by differentiating the action with respect to V_1 and V_2. We get

$$V_1^\dagger (W - W^\dagger) V_1 - (W - W^\dagger) = 0 \qquad \text{and} \qquad V_2^\dagger (W - W^\dagger) V_2 - (W - W^\dagger) = 0\,, \quad (4.4)$$

where

$$W = e^{-2\pi i m/n} V_1 V_2 V_1^\dagger V_2^\dagger\,. \qquad (4.5)$$

These two equations possess a simple geometrical interpretation: the difference $(W - W^\dagger)$, which in the large-N limit is related to the field strength of the noncommutative theory, is covariantly constant when we move in the direction 1 or in the direction 2. In the matrix language the above equations simply assert that the difference $(W - W^\dagger)$ commutes with the unitary matrices V_a, $a = 1, 2$.

To find the general solutions of these equations we have found useful to use the projectors technique: the first step is to introduce the spectral decomposition of the matrix W

$$W = \sum_j e^{i\phi_j} P_j \,. \tag{4.6}$$

Here P_j is the orthogonal projector on the $j-$th eigenspace of W and $\exp(i\phi_j)$ is its eigenvalue. They satisfy the properties

$$P_k P_j = \delta_{kj} P_j \qquad \text{and} \qquad \sum_j P_j = \mathbb{I} \,. \tag{4.7}$$

We recall that this decomposition exists since this matrix is unitary. In the following, however, the relevant combination is only the difference $W - W^\dagger$, whose spectral decomposition is then

$$W - W^\dagger = 2i \sum_j \sin\phi_j P_j \,. \tag{4.8}$$

We must notice that eigenspaces corresponding to different eigenvalues for the matrix W can merge and correspond to the same eigenvalue for the above difference. This occurs when the matrix W possesses both the eigenvalue $\exp(i\phi_j)$ and the eigenvalue $\exp(i(\pi - \phi_j))$. In fact, it is easy to see that they produce the same eigenvalue for the difference $W - W^\dagger$ and thus their eigenspaces coalesce. For this reason, we shall rewrite eq. (4.8) as the reduced sum

$$W - W^\dagger = 2i \sum_\ell \sin\phi_\ell \mathbb{P}_\ell \,, \tag{4.9}$$

where $\mathbb{P}_\ell$ are the orthogonal projectors on the eigenspaces of the anti-hermitian matrix $W - W^\dagger$ and they satisfy the analogous of eq. (4.7). We have also that $\mathbb{P}_\ell = P_\ell$ if $\exp(i\phi_\ell)$ is an eigenvalue of W, but $\exp(i(\pi - \phi_\ell))$ is not. If they are both eigenvalues and $P_{1\ell}$ and $P_{2\ell}$ are the corresponding projectors, $\mathbb{P}_\ell = P_{1\ell} + P_{2\ell}$.

The equations of motion are now equivalent to the fact the matrices V_a commute with the projectors $\mathbb{P}_j$, namely

$$\mathbb{P}_j V_a = V_a \mathbb{P}_j \,. \tag{4.10}$$

Let us now define the reduced matrices

$$\mathbb{V}_a^{(j)} = \mathbb{P}_j V_a \mathbb{P}_j \,, \tag{4.11}$$

then along the equations of motion

$$V_a = \sum_j \mathbb{V}_a^{(j)} = \sum_j \mathbb{P}_j V_a \mathbb{P}_j \,. \tag{4.12}$$

In other words, the matrices V_a can be put in a block-diagonal form since the eigenspaces defined by the projectors $\mathbb{P}_j$ are invariant subspaces. The reduced matrices $\mathbb{V}_a^{(j)}$ are also unitary on their eigenspace,

$$\mathbb{V}_a^{(j)\dagger} \mathbb{V}_a^{(j)} = \mathbb{P}_j V_a^\dagger \mathbb{P}_j^2 V_a \mathbb{P}_j = \mathbb{P}_j V_a^\dagger V_a \mathbb{P}_j^3 = \mathbb{P}_j^4 = \mathbb{P}_j \,. \tag{4.13}$$

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Since the matrices V_a are block-diagonal the equations of motion are trivially satisfied. Actually, we have still to impose that our spectral decomposition holds or equivalently in each subspace we must require

$$\mathbb{P}_j(W - W^\dagger)\mathbb{P}_j = 2i \sin\phi_j \mathbb{P}_j \,. \tag{4.14}$$

This relation does not hold automatically because the matrix W depends on the matrices V_a and it gives a constraint on the form of the matrices V_a in each subspace. It can be rewritten in terms of reduced matrices eq. (4.11). Namely, if we define

$$\mathbb{W}^{(j)} = e^{-2\pi i m/n} \mathbb{V}_1^{(j)} \mathbb{V}_2^{(j)} \mathbb{V}_1^{(j)\dagger} \mathbb{V}_2^{(j)\dagger} \tag{4.15}$$

the above equation reads

$$\mathbb{W}^{(j)} - \mathbb{W}^{(j)\dagger} = 2i \sin\phi_j \mathbb{P}_j \,. \tag{4.16}$$

At this point we simply have the two possibility already discussed:

- *type-I:* $\exp(i\phi_j)$ is an eigenvalue of W, but $\exp(i(\pi - \phi_j))$ is not. Then $\mathbb{P}_j = P_j$ and we can write

  $$\mathbb{W}^{(j)} = e^{-2\pi i m/n} \mathbb{V}_1^{(j)} \mathbb{V}_2^{(j)} \mathbb{V}_1^{(j)\dagger} \mathbb{V}_2^{(j)\dagger} = \exp(i\phi_j) P_j \,. \tag{4.17}$$

  The value of $\exp(i\phi_j)$ can be now determined by taking the determinant of both sides. We have

  $$\exp\left(-2\pi i n_j \frac{m}{n}\right) = \exp(i n_j \phi_j) \quad \Rightarrow \quad \phi_j = 2\pi \left(\frac{m_j}{n_j} - \frac{m}{n}\right), \tag{4.18}$$

  where n_j is the dimension of the subspace and m_j is an integer number that runs from 0 to $n_j - 1$. Now eq. (4.17) on the restricted subspace is

  $$\mathbb{V}_1^{(j)} \mathbb{V}_2^{(j)} = \exp\left(2\pi i \frac{m_j}{n_j}\right) \mathbb{V}_2^{(j)} \mathbb{V}_1^{(j)} \,. \tag{4.19}$$

  This is exactly the equation for the twist-eaters of general twist, whose solutions are widely discussed in the literature. We remark that, at this level, fixed the dimension n_j of the subspace we have a different solution for any choice of m_j: this is not the end of the story because m_j and n_j are not coprime, in general, and therefore the space of the solutions needs a more refined treatment. When $\gcd(m_j, n_j) = \hat{n}_j \neq 1$, the representation of the Weyl-'t Hooft algebra eq. (4.19) is no longer irreducible, having $\hat{n}_j$ irreducible components. It means that, up to gauge transformations, we can always take $\mathbb{V}_1^{(j)}, \mathbb{V}_2^{(j)}$ to be block diagonal, the $\hat{n}_j$ blocks being irreducible: we could expect naively a solutions space

  $$M_{(m_j, n_j)} = (\tilde{T}^2)^{\hat{n}_j} = \tilde{T}^2 \times \tilde{T}^2 \times \cdots \times \tilde{T}^2 \,, \tag{4.20}$$

  labelling the freedom in choosing the U(1) phases of the $\hat{n}_j$ blocks. We have instead to consider also the residual gauge symmetry which acts by permuting the $\hat{n}_j$ irreducible components: this action, on the solutions space eq. (4.20), is represented by the permutation group $S_{\hat{n}_j}$, and therefore the moduli space is the symmetric orbifold [26]

$$\mathcal{M}_{(m_j,n_j)} = \mathrm{Sym}^{\hat{n}_j}\hat{T}^2 = \frac{(\tilde{T}^2)^{\hat{n}_j}}{S_{\hat{n}_j}}\,. \tag{4.21}$$

We have seen that the general solution of the equations of motions is always block-diagonal: we can evaluate directly, therefore, the contribution of the $j-$th block to the classical action. The relevant quantity is $\mathrm{tr}(\mathbb{W}^{(j)})+\mathrm{tr}(\mathbb{W}^{(j)\dagger})$ and, happily, it can be computed without an explicit knowledge of the form of the solution. In fact we have

$$\mathrm{tr}(\mathbb{W}^{(j)})+\mathrm{tr}(\mathbb{W}^{(j)\dagger}) = 2n_j\cos\phi_j = 2n_j\cos\left(2\pi\left(\frac{m_j}{n_j}-\frac{m}{n}\right)\right). \tag{4.22}$$

Let us notice that the explicit form of the original twist enters, in our procedure, directly at level of classical action.

- *type-II:* both $\exp(i\phi_j)$ and $\exp(i(\pi-\phi_j))$ are an eigenvalues of W. Denoting with P_{1j} and P_{2j} the corresponding projectors, we have

$$\mathbb{W}^{(j)} = e^{-2\pi i m/n}\mathbb{V}_1^{(j)}\mathbb{V}_2^{(j)}\mathbb{V}_1^{(j)\dagger}\mathbb{V}_2^{(j)\dagger} = \exp(i\phi_j)P_{1j} - \exp(-i\phi_j)P_{2j}\,. \tag{4.23}$$

We have now to find two matrices that satisfies these equation. Again, taking the determinant (on this subspace) of both sides, we get an equation

$$\exp(-2\pi i\left(d_{1j}+d_{2j}\right)\frac{m}{n}\Big) = \exp(i\phi_j d_{1j} + i d_{2j}(\pi-\phi_j))\,, \tag{4.24}$$

that relates the parameters ϕ_j to the dimensions d_{1j} and d_{2j} of the subspaces. The existence of this kind of extrema was noticed in ref. [47] and it was not clear if they could play some role in the large-N limit: it was suggested that they are the analog of multi-instantons configurations. We will show, in the next section, that they are suppressed here as N becomes large. To this aim we have to first evaluate their block-contributions to the classical action. Even in this case, the sum $\mathrm{tr}(\mathbb{W}^{(j)})+\mathrm{tr}(\mathbb{W}^{(j)\dagger})$ can be easily computed and one finds

$$\mathrm{tr}(\mathbb{W}^{(j)})+\mathrm{tr}(\mathbb{W}^{(j)\dagger}) = 2(d_{1j}-d_{2j})\cos\phi_j\,. \tag{4.25}$$

The most general solution of the TEK model can be built, therefore, by considering the direct sum of solutions of type-I and type-II. It is quite clear that to any partition of n we can associate, in principle, a solution: be $\{\nu_j\}=\{\nu_1,\nu_2,\ldots,\nu_n\}$ the natural numbers describing the partition

$$n = \nu_1 + 2\nu_2 + \cdots + n\nu_n\,. \tag{4.26}$$

We can associate to $\{\nu_j\}$ a subspaces decomposition with ν_1 one-dimensional subspaces ($d_1=1$), ν_2 two-dimensional subspaces ($d_2=2$) and so on. On any of these subspaces we can have or solutions of type-I or solutions of type-II. For type-I solutions we have been able to associate another integer, m_j, allowing for an explicit realization in terms of twist-eaters , while for type-II we did not succed in giving such a simple description. This is not the end of the story, of course, we still have the freedom to weight any subspace solutions with an arbitrary U(1) phase: more importantly we have to refine the relation between partitions and distinct solutions. In fact it is not difficult to show that, for type-I configurations, different choices of (n_j, m_j) result in the same solution. We will discuss the necessary refinement in the next section, in connection also with the moduli space structure of type-I family.

5. Solutions of finite action in the double scaling limit

In the previous section we have solved the classical equations of motion for the two-dimensional TEK model: in order to recover the noncommutative gauge theory we have still to perform the double scaling limit. It is well known [20, 21, 22] that on the noncommutative plane finite action solutions exist and we have shown in [17] that the partition function and Wilson lines correlators seems to be localized around them. More recently [23, 27] the same properties have been shown to hold even at finite volume, where a localization theorem has been proven and an explicit form of the partition function has been proposed. We expect therefore that some of our TEK classical solutions survive the relevant double scaling limits (i.e. their classical action remain finite) and reproduce at least some instanton-like feature.

Let ℓ be the index running over the solutions of type-I and n_ℓ the corresponding dimensions. In the same way let s be the index spanning the solutions of type-II and D_{1s} and D_{2s} the dimensions of the associated subspaces. Then the value of the action on a classical solution is

$$S_{TEK} = \beta N \left(2n - 2\sum_\ell n_\ell \cos\phi_\ell - 2\sum_s (D_{1s} - D_{2s}) \cos\phi_m \right) . \tag{5.1}$$

Recalling the sum rule

$$\sum_\ell n_\ell + \sum_s (D_{1s} + D_{2s}) = n\,, \tag{5.2}$$

the above equation can be rewritten as a sum positive definite objects

$$S_{TEK} = \beta N \left(\sum_\ell n_\ell \sin^2\left(\frac{\phi_\ell}{2}\right) + \sum_s \left(D_{1s} \sin^2\left(\frac{\phi_s}{2}\right) + D_{2s} \cos^2\left(\frac{\phi_s}{2}\right)\right)\right) . \tag{5.3}$$

In the following we shall look for solutions such that S_{TEK} is finite in the large-N limit: we have in mind, of course, to send also n to infinity, according to the precise scalings derived in section 3. Since the overall coefficient in eq. (5.3) diverges in this limit, both in recovering the NC torus and the NC plane, this occurs only when each term in the quantity between parenthesis goes to zero quickly enough. Namely, we must have

$$n_\ell \sin^2\left(\frac{\phi_\ell}{2}\right) \to 0 \tag{5.4}$$

and

$$D_{1s}\sin^2\left(\frac{\phi_s}{2}\right)\to 0 \qquad D_{2s}\cos^2\left(\frac{\phi_s}{2}\right)\to 0\,. \tag{5.5}$$

First, we examine eq. (5.5): if both D_{1s} and D_{2s} are different from zero, this condition cannot be satisfied. In fact it would require that the sine and the cosine of the same angle vanish. If one of the two dimensions is zero, the solution of type-II collapses in a solution of type-I. This case is impossible because we assume to have already counted all the solutions of type-I in the first sum. So the only possibility left is

$$D_{1s}=D_{2s}=0 \qquad \forall\, s\,. \tag{5.6}$$

In other words, solutions of type-II have not finite action in the large-N limit: we are left with the solution of type-I and the value of S_{TEK} is

$$S_{TEK}=\beta N\sum_\ell n_\ell\sin^2\pi\left(\frac{m_\ell}{n_\ell}-\frac{m}{n}\right), \tag{5.7}$$

where

$$\sum_\ell n_\ell=n \qquad \text{and} \qquad m_\ell=0,\dots,n_\ell-1\,. \tag{5.8}$$

In order to have finite limit, each term in the sum eq. (5.7) must be finite, since no cancellation can intervene among different terms, being in fact all positive. We start by discussing the noncommutative torus.

5.1 Solutions with finite action on the torus

Let us consider the general case with p and q coprime: m/n is given by eq. (2.47) and the dimension of the matrices is $Np-2rq$. We have to impose the scaling on β: as we have already discussed, we choose

$$\beta=\frac{2}{g^2a^2}=\frac{2N^2}{g^2A}\,, \tag{5.9}$$

to perform the double scaling limit. In fact, the coupling scales canonically with the lattice spacing a, namely $\beta\propto a^{-2}\propto N^2$. Here A denotes the area of the noncommutative torus and g^2 is the physical coupling constant, the factor 2 being inserted for later convenience. Only the type-I solutions can have a finite limit and the value of the action is

$$S_{TEK}=\frac{2}{g^2A}N^3\sum_\ell n_\ell\sin^2\left(\frac{\phi_\ell}{2}\right)=\frac{2}{g^2A}N^3\sum_\ell n_\ell\sin^2\pi\left(\frac{m_\ell}{n_\ell}-\frac{m}{n}\right), \tag{5.10}$$

where

$$\sum_\ell n_\ell=Np-2rq \qquad \text{and} \qquad m_\ell=0,\dots,n_\ell-1\,. \tag{5.11}$$

To perform the limit we use the parametrization

$$n_\ell=Np_\ell-(2r)q_\ell \qquad m_\ell=-sp_\ell+kq_\ell\,, \tag{5.12}$$

to rewrite the action in the form

$$S_{TEK} = \frac{2}{g^2 A} N^4 \sum_{\ell} (p_\ell - \theta q_\ell) \sin^2 \pi \left(\frac{-sp_\ell + kq_\ell}{Np_\ell - (2r)q_\ell} - \frac{-sp + kq}{Np - (2r)q} \right)$$
$$= \frac{2}{g^2 A} N^4 \sum_{\ell} (p_\ell - \theta q_\ell) \sin^2 \frac{\pi}{N^2} \left(\frac{q_\ell}{p_\ell - \theta q_\ell} - \frac{q}{p - \theta q} \right). \tag{5.13}$$

At this point we take the large-N limit, assuming that $\theta = 2r/N$ approaches an irrational number: we easily get

$$S_{TEK} = \frac{2\pi^2}{g^2 A} \sum_{\ell} (p_\ell - \theta q_\ell) \left(\frac{q_\ell}{p_\ell - \theta q_\ell} - \frac{q}{p - \theta q} \right)^2. \tag{5.14}$$

This is the central result of the paper: eq. (5.14) exactly reproduces the value of the action of Yang-Mills theory on a (p, q) projective module, evaluated on the classical solutions of the equations of motion [23, 26]. To get a complete identification we have to specify the range of p_ℓ, q_ℓ and to show they satisfy indeed the correct constraints.

The first thing to notice is that the positivity of the dimensions implies

$$Np - 2rq > 0 \quad \Rightarrow \quad p - \theta q > 0$$
$$n_\ell > 0 \quad \Rightarrow \quad p_\ell - \theta q_\ell > 0\,. \tag{5.15}$$

Then we have to establish a sum rule over p_ℓ and q_ℓ: one would be tempted to set the sums equal to p and q, namely

$$\sum_{\ell} q_\ell = q \qquad \sum_{\ell} p_\ell = p\,. \tag{5.16}$$

This cannot be done in a straightforward manner. On p_ℓ and q_ℓ we have only the constraint $\sum_\ell n_\ell = Np - 2rq$, which in turn implies

$$\sum_{\ell} q_\ell = q - Nj \qquad \sum_{\ell} p_\ell = p - 2rj\,. \tag{5.17}$$

However if $2r/N$ approaches an irrational number the only consistent choice in the large-N limit is $j = 0$. This gives the desired sum rules for p_ℓ and q_ℓ, establishing the connection with the "partitions" described in [23, 26]. The total moduli space is also seen to be trivially the same, in the limit we are considering: following [23], a given partition (p_ℓ, q_ℓ), compatible with eq. (5.15), can be unambiguously presented as $(N'_\ell, p'_\ell, q'_\ell)$ with p'_ℓ, q'_ℓ relatively prime and N'_ℓ is the associated multiplicity. From eq. (5.12) we obtain, at level of TEK solutions, the related presentation as $(N'_\ell, m'_\ell, n'_\ell)$: the integers N'_ℓ simply count the number of irreducible representations of a given Weyl-'t Hooft algebras (labelled by the relatively prime integers (m'_ℓ, n'_ℓ)) inside a given solution and they are enough to construct the total moduli space

$$\mathcal{M}_{(m,n)} = \Pi_\ell \mathcal{M}_{(N'_\ell m'_\ell, N'_\ell n'_\ell)} = \Pi_\ell \mathrm{Sym}^{N'_\ell} \tilde{T}^2\,, \tag{5.18}$$

the ℓ index running over the same partitions defined by eq. (5.15). Once the relation between $(N'_\ell, n'_\ell, m'_\ell)$ and $(N'_\ell, p'_\ell, q'_\ell)$, as determined by eq. (5.12) in the large-N limit, is realized, we have the full identification of the moduli space of classical solutions of Yang-Mills theory on the (p, q) projective module with the moduli space of our finite action TEK classical solution

$$\mathcal{M}_{(p,q)} = \mathcal{M}_{(m,n)} \,. \tag{5.19}$$

The emerging of these finite action configurations crucially relies on the scaling eq. (5.9) and on the judicious subtractions performed in defining the starting TEK action. The first one, appearing in eq. (2.58), makes the action always positive and it corresponds to the subtraction of a divergent term, in the large-N limit, independent of the details of the particular projective module. It is related to the zero-point energy of the system and it usually appears also in conventional lattice theory. The second subtraction is subtler and it is performed by fixing the background connection in such a way that the absolute minimum of the action, in a given (p, q) sector, is at zero value, for the constant curvature connection characterizing the projective module itself. This choice produces a delicate cancellation between the difference inside the sine in eq. (5.13), resulting into the $1/N^2$ decaying of the argument instead of a generic $1/N$. Conversely, after taking the limit, we obtain the classical action in presence of a background connection, having minimum zero at the correct value (see [23] for a discussion of the subtraction on the continuum). It could be nevertheless useful to choose a different value of the continuum background connection: in particular, we can fix the minimum in such a way that eq. (5.14) reproduces the commutative value of the action, for a given Chern class, in the limit $\theta \to 0$. We modify therefore eq. (2.64) as

$$\phi \to \phi + i\pi \frac{q}{nN} = \phi + \frac{i\pi}{N^2}\frac{q}{p - \theta q} \ : \tag{5.20}$$

it is simple to verify that eq. (5.14) changes as

$$S_{TEK} = \frac{2\pi^2}{g^2 A} \sum_\ell \left(\frac{q_\ell^2}{p_\ell - \theta q_\ell} \right). \tag{5.21}$$

Let us notice that the modification we have done is of order $1/N^2$, preserving in this way the delicate balance we have observed to produce the finite action solutions. Nicely, as $\theta \to 0$, we can recover the classical configurations solving the commutative theory: for $p = 1$ we have the unique choice $q_\ell = q$, giving us the U(1)-instanton associated to the Chern number q. For general p we obtain the solutions of the U(p) theory in the relevant charge sector. Having recovered the classical solutions of the commutative theory for finite rank of the gauge group from a one-plaquette model is not trivial at all: in fact, as we will discuss in section 6, the exact solution of the familiar YM_2 involves a different kind of lattice discretization.

We see that eq. (5.14) confirms our interpretation of the parameter q as the Chern class associated to the module. However it would be interesting to have a direct computation of this quantity as the discretized integral of the lattice field strength. The equations of motions and a simple analysis for small lattice spacing suggest to identify the lattice field strength with

$$F = \frac{i}{2}\Big(D_1 D_2 D_1^\dagger D_2^\dagger - D_2 D_1 D_2^\dagger D_1^\dagger \Big) \,, \tag{5.22}$$

in the large-N limit. Then its trace (the integral in the noncommutative language) produces

$$Q = \frac{i}{2}\,\mathrm{Tr}\Big[D_1 D_2 D_1^\dagger D_2^\dagger - D_2 D_1 D_2^\dagger D_1^\dagger\Big]\,. \tag{5.23}$$

We have

$$\begin{aligned} Q &= \frac{i}{2} N \sum_\ell \left[n_\ell \exp\left(\frac{2\pi i s}{N}\right) \exp\left(\frac{2\pi i (k q_\ell - p_\ell s)}{n_\ell}\right) - \text{c.c.}\right] \\ &= N^2 \sum_\ell (p_\ell - \theta q_\ell) \sin\left(\frac{2\pi q_\ell}{N^2 (p_\ell - q_\ell \theta)}\right), \end{aligned} \tag{5.24}$$

that in the large-N limit gives us the desired relation

$$Q = 2\pi \sum_\ell q_\ell = 2\pi q\,. \tag{5.25}$$

5.2 Solutions with finite action on the plane

Reaching the noncommutative plane is actually tricker than the torus. We recall, in fact, that the only relevant finite quantity in the large-N limit is the dimensional combination

$$\Theta_{\text{plane}} = \frac{\theta A}{2\pi} = N^2 a^2 \frac{2r}{2N\pi} = \frac{r N a^2}{\pi}\,. \tag{5.26}$$

Its finiteness, in turn, implies that we can drop the dependence on the area in all our expressions in favor of that on Θ_{plane},

$$A = \frac{\pi N \Theta_{\text{plane}}}{r}\,. \tag{5.27}$$

Then the value of the S_{TEK} action eq. (5.13) is more conveniently written as

$$S_{TEK} = \frac{2r}{\pi g^2 \Theta_{\text{plane}}} N^3 \sum_\ell (p_\ell - \theta q_\ell) \sin^2 \frac{\pi}{N^2}\left(\frac{q_\ell}{p_\ell - \theta q_\ell} - \frac{q}{p - \theta q}\right). \tag{5.28}$$

Two remarks immediately stem from a first inspection of eq. (5.28): the factor in front of the action scales as N^3 (and not as N^4) and the parameter θ is no longer a finite quantity in the large-N limit, but its values flows to zero as $1/N$. Thus, differently from what happened in the torus case, each term in the sum behaves as $1/N$ and it possesses a vanishing limit

$$\begin{aligned} \frac{2rN^3(p_\ell - \theta q_\ell)}{\pi g^2 \Theta_{\text{plane}}} \sin^2 \frac{\pi}{N^2}\left(\frac{q_\ell}{p_\ell - \theta q_\ell} - \frac{q}{p - \theta q}\right) &\simeq \frac{2r\pi^2 N^3 (p_\ell - \theta q_\ell)}{\pi g^2 \Theta_{\text{plane}} N^4}\left(\frac{q_\ell}{p_\ell - \theta q_\ell} - \frac{q}{p - \theta q}\right)^2 \\ &\simeq \frac{2\pi r (p_\ell - \theta q_\ell)}{g^2 \Theta_{\text{plane}} N}\left(\frac{q_\ell}{p_\ell - \theta q_\ell} - \frac{q}{p - \theta q}\right)^2 \longrightarrow 0\,. \end{aligned} \tag{5.29}$$

The classical solutions of the TEK model in the large-N limit defining the plane appear to form a democratic sea of vanishing action configurations. No sign of the fluxons discussed in [20, 22] is apparently present.

Fortunately, this way of reasoning has a welcome exception when some of the p_l in the partition of p vanish. When $N \to \infty$, the corresponding term in the action takes the form

$$\begin{aligned}\frac{2rN^3(-\theta q_\ell)}{\pi g^2\Theta_{\rm plane}}\sin^2\frac{\pi}{N^2}\left(\frac{1}{\theta}+\frac{q}{p-\theta q}\right) &\simeq \frac{2rN^3(-\theta q_\ell)}{\pi g^2\Theta_{\rm plane}}\frac{\pi^2}{N^4\theta^2}\left(1+\frac{q\theta}{p-\theta q}\right)^2 \\ &\simeq -\frac{2\pi r q_\ell}{g^2\Theta_{\rm plane}N\theta}\left(1+\frac{q\theta}{p-\theta q}\right)^2 \longrightarrow -\frac{\pi q_\ell}{g^2\Theta_{\rm plane}}\,,\end{aligned} \tag{5.30}$$

where we have used that $N\theta = 2r$. The limit is totally independent of the free parameter r that we had at the beginning in the definition of $\Theta_{\rm plane}$. Collecting the different contributions, the total action is then

$$S_{\rm plane} = -\sum_{\hat\ell}\frac{\pi q_{\hat\ell}}{g^2\Theta_{\rm plane}} = -\frac{\pi\sum_{\hat\ell}q_{\hat\ell}}{g^2\Theta_{\rm plane}} \equiv -\frac{\pi\hat k}{g^2\Theta_{\rm plane}}\,, \tag{5.31}$$

where the sum runs only over the elements of the partition with vanishing p_ℓ. This is exactly the value of the action for the fluxons discovered by [20, 22]. In fact it exhibits the two peculiar features of these topological objects, solving the classical Yang-Mills equations on the noncommutative plane: its value grows linearly with the topological charge at variance with quadratic behavior for the instanton solutions on the torus. Moreover the sign of total charge $\hat k$ obeys the positivity constraint

$$-\Theta_{\rm plane}\hat k > 0\,, \tag{5.32}$$

which simply asserts that the sum of the dimensions of the eigenspaces contributing is positive. This also displays the chiral nature of these solutions.

The picture emerging from the above limiting procedure deserves some comments. First of all, we notice that the same value of the action can be obtained starting from very different configurations at the level of the matrix model: the only necessary ingredient is the presence of a certain number of vanishing p_l whose corresponding q_ℓ sum to the desired total charge $\hat k$. The possible choices of the relevant q_ℓ are therefore in correspondence with the partitions of $|\hat k|$. This inner structure should have a natural interpretation in terms of the moduli space of fluxons [48].

There is another subtle source of degeneracy, already announced at the beginning of this section: the subset of the partition with non vanishing p_ℓ is more or less unconstrained, since it gives a vanishing contribution to the total action. The only conditions that the p_ℓ must fulfill are

$$\sum_\ell p_\ell = p \qquad \text{with } p_\ell > 0\,, \tag{5.33}$$

while the corresponding q_ℓ sum to $q-\hat k$. The positivity of the p_ℓ is what survives of eq. (5.15) in the large-N limit. It is worth noticing that the additional structure generated by the p_ℓ strongly resembles that of a commutative instanton of $\mathrm{U}(p)$ with Chern class equal to $q-\hat k$ on the torus and thus it seems to carry all the original geometrical data. It is also well-known that these commutative configurations become degenerate with the vacuum in the decompactification limit, as it happens here. Notice that the above degeneracy disappear for $p = 1$ and the large area limit can be carried in straightforward manner safely reaching the so-called U(1) Yang-Mills theory on the noncommutative plane. This was also the choice made in ref. [17].

The exact correspondence between fluxons with their ancestors on the noncommutative torus as well as their contribution to the partition function on the plane will be discussed in details in ref. [48].

6. A few remarks on the quantum theory

The success in describing the instantons over the noncommutative torus strongly suggests that a complete quantum analysis of NCYM_2 may be not out of reach in our framework. This hope is also corroborated by the observation that NCYM_2 is expected to be semi-classically exact [23]: namely the quantum observables can be expressed as a sum over the classical solutions, weighted with a fluctuations factor.

However the situation is more intricate than one can naively think. The path that has led Migdal to solve the usual QCD_2 encounters here a couple of serious obstacles: the continuum limit is, as already stated many times, intrinsically tied with a large-N limit, while in the ordinary theory it can be safely taken at finite N. Second, a group theory characterization of the emerging theory is quite difficult, the representation theory of $\mathrm{U}(\infty)$ being almost unknown at variance with that of $\mathrm{U}(N)$.

Let us start by discussing some general facts: the computation of the partition function,

$$\mathcal{Z}_{TEK} = \int \frac{DV_1 DV_2}{\mathrm{Vol}(\mathrm{U}(n))} \exp\left(\beta N \,\mathrm{Tr}\left[e^{2i\pi\frac{m}{n}} V_1 V_2 V_1^\dagger V_2^\dagger + e^{-2i\pi\frac{m}{n}} V_2 V_1 V_2^\dagger V_1^\dagger\right] - 2\beta N n\right) \quad (6.1)$$

can be reduced to a one-matrix integral through two steps. To show this we first introduce the following representation of delta function over the unitary group in terms of the characters χ_R

$$\sum_R \chi_R(V_1 V_2 V_1^\dagger V_2^\dagger)\chi_R(W^\dagger) = \delta(V_1 V_2 V_1^\dagger V_2^\dagger, W)\,, \quad (6.2)$$

with the sum running over all representations of $\mathrm{U}(n)$, and then we perform the integral over V_i by means of the formula

$$\int DU\, \chi_R(UAU^\dagger B) = \frac{\chi_R(A)\chi_R(B)}{d_R}\,. \quad (6.3)$$

The final result is

$$\mathcal{Z}_{TEK} = \int \frac{DW}{\mathrm{Vol}(\mathrm{U}(n))} \exp\left(\beta N \left[\mathrm{Tr}(W) + \mathrm{Tr}(W^\dagger)\right] - 2\beta N n\right) \sum_R \frac{\chi_R(e^{-2i\pi\frac{m}{n}} W^\dagger)}{d_R}\,. \quad (6.4)$$

We can even compute the integral over W using the formula

$$\lambda_R(\beta N) = \frac{1}{d_R} \int \frac{DW}{\mathrm{Vol}(\mathrm{U}(n))} \exp\left(\beta N \left[\mathrm{Tr}(W) + \mathrm{Tr}(W^\dagger)\right]\right) \chi_R(W) \quad (6.5)$$

where $\lambda_R(\beta N)$ can be explicitly expressed through the determinant of a matrix having Bessel functions as entries [49], whose form will be not relevant for our discussion. Accordingly the partition function takes the compact form

$$\mathcal{Z}_{TEK} = \sum_R \lambda_R \left(\frac{2N^3}{g^2 A} \right) \exp \left(-2\pi i n_R \frac{m}{n} - \frac{4N^3 n}{g^2 A} \right), \tag{6.6}$$

where n_R is the total number of boxes in the Young tableaux associated the representation R and we have also used that $\beta = 2N^2/g^2 A$ for the noncommutative torus. The other term in the exponential is, of course, the subtraction introduced in section 2. The apparent simplicity of this expression is misleading, hiding the complexity of the double scaling limit necessary to reach the continuum. To better understand the roots of the difficulties that one would encounter as $N \to \infty$, it is instructive to contrast this expression with the usual QCD_2 on the lattice

$$\mathcal{Z}_{QCD_2} = \sum_R \left(\lambda_R \left(\frac{2s^2}{g^2 A} \right) \right)^{s^2} \exp \left(-\frac{4Ns^4}{g^2 A} \right), \tag{6.7}$$

where s^2 is the total number of the sites. We recall that here N has to be kept fixed while s goes to infinity. In this limit the combination

$$\left(\lambda_R \left(\frac{2s^2}{g^2 A} \right) \exp \left(-\frac{4Ns^2}{g^2 A} \right) \right)^{s^2}$$

can be nicely evaluated by using the asymptotic expansion of the Bessel functions, appearing in the explicit expression of λ_R: up to an irrelevant multiplicative overall constant, the leading contribution exponentiates producing the celebrated result [19]

$$\mathcal{Z}_{QCD_2} = \sum_R \exp \left(-\frac{g^2 A}{2} C_2(R) \right), \tag{6.8}$$

$C_2(R)$ being the quadratic Casimir of the representation R. In spite of their superficial similarity, there are both technical and conceptual differences between eq. (6.6) and eq. (6.7). For example, in the second case, the sum over group representation is a spectator in the continuum limit (large s), which in turn can be taken term by term. Viceversa in the first case, where the continuum limit is at large-N (large group rank), the number of integers you sum over keeps growing. It is also difficult, in this case, to isolate the asymptotic behavior of each term in the sum, since the determinant of Bessel functions defining λ_R depends on N both through the form of its matrix elements and through its dimension. These different behaviors reflect deeply two alternative ways to deal with space-time discretization: the usual approach, where the lattice is an entity independent of its matter content contrasted with the noncommutative point of view, where space-time and gauge symmetries are indissolubly tied.

An attempt to investigate in a concrete way the large-N limit of $\mathcal{Z}_{TEK}$ was performed in [50], where the partition function has been expressed through an expansion in inverse powers of the coupling constant: unfortunately the authors have not been able to discuss the complexity of the double scaling limit. We shall not try to tackle here this difficult problem, which would require a deeper understanding on how classical configuration should dominate the large-N limit: recall, in fact, that the theory is believed to be semiclassically exact. However a simple computation, checking the consistency of the TEK approach at quantum level, can be performed in a quite straightforward manner. Let us consider in fact the following observables

$$\mathcal{O}_k = \frac{1}{n\mathcal{Z}_{TEK}} \int \frac{DV_1\, DV_2}{\mathrm{Vol}(\mathrm{U}(n))} \exp(-S_{TEK})\, \mathrm{Tr}\left[\left(V_1 V_2 V_1^\dagger V_2^\dagger\right)^k\right]. \tag{6.9}$$

It is not difficult to show that O_k is the quantum average of a Wilson loop winding k-times around the fundamental plaquette, the area surrounded by the contour being a^2 (we recall that a is the lattice spacing). We want to compute this object in a particular limit: when setting $a \to 0$, to reach the continuum limit, we also consider the winding k very large so that

$$k^2 a^2 = \lambda\,. \tag{6.10}$$

Following closely what we have done above, we can reduce the computation to the one-matrix integral

$$\begin{aligned}\mathcal{O}_k =&\ \frac{1}{n\mathcal{Z}_{TEK}} \int \frac{DW}{\mathrm{Vol}(\mathrm{U}(n))} \times \\ &\times \exp\left(\beta N\left[\mathrm{Tr}(W) + \mathrm{Tr}(W^\dagger)\right] - 2\beta N n\right) \mathrm{Tr}[W^k] \sum_R \frac{\chi_R(e^{-2i\pi\frac{m}{n}} W^\dagger)}{d_R}\,.\end{aligned} \tag{6.11}$$

In the following we shall restrict our investigation to the two-dimensional noncommutative plane (we choose for simplicity $r = 1$ and $N = n$): $\mathcal{O}_k$, in this case, corresponds to a Wilson loop average of vanishing area, winding an infinite number of times. On the commutative plane observables of this type have been considered in [51]: there it has been observed, starting from the exact solution at finite k and non-vanishing area, that this particular limit correspond to a truly perturbative situation, being completely determined by the zero-instanton approximation on compact surfaces. We expect, therefore, that also in the noncommutative case these observables can be captured by a perturbative computation.

By exploiting the double scaling limit relevant for the plane we obtain a simple relation between k and n in the form

$$k^2 = \frac{\pi\lambda}{\Theta} n\,. \tag{6.12}$$

Let us now evaluate eq. (6.11) in the limit of large k: we expect, in this case, that the integral is dominated by the classical solution $W = \mathbb{1}$. We write

$$W = \exp(i\hat{H}) \tag{6.13}$$

where $\hat{H}$ is an hermitian matrix: we see that the rapid oscillations of the holonomy factor are controlled by redefining $\hat{H} = H/k$, forcing therefore an expansion around $W = \mathbb{1}$. Within this approximation O_k can be expressed as an integral over the hermitian matrix H

$$\mathcal{O}_k = \frac{1}{\mathcal{Z}_{TEK} n} \sum_R \frac{e^{-2i\pi n_R \frac{m}{n}}}{k^{n^2/2}} \left[\int \frac{DH}{\mathrm{Vol}(\mathrm{U}(n))} \exp\Big(-\frac{\beta n}{k^2}\,\mathrm{Tr}[H^2] + iH\Big) + o\left(\frac{1}{k}\right)\right]. \tag{6.14}$$

The combination $\beta n/k^2$ turns out to be

$$\frac{\beta n}{k^2}=\frac{2n}{g^2\lambda}\,. \tag{6.15}$$

By computing $\mathcal{Z}_{TEK}$ in the same approximation and then using standard technique to evaluate the matrix integral we arrive at

$$\mathcal{O}_k=W_n(\lambda)=\frac{1}{n}\exp\left(-\frac{g^2\lambda}{4n}\right)L^1_{n-1}\left(\frac{g^2\lambda}{8n}\right), \tag{6.16}$$

expressed through the Laguerre polynomial $L^1_m(z)$. The limit $n\to\infty$ can be now easily obtained

$$W_n(\lambda)\to 2\sqrt{\frac{2}{\lambda g^2}}J_1\left(\sqrt{\frac{\lambda g^2}{2}}\right), \tag{6.17}$$

$J_1(z)$ being a Bessel function. The above result deserves some considerations: first of all the computation we did it is essentially "perturbative", although justified by the large k-behavior. We have, in fact, perturbed the path-integral around the classical solution $W=\mathbb{1}$, that should correspond to the usual perturbative expansion on the noncommutative plane. Secondly, we notice that the final result is independent from the effective Θ parameter on the plane, suggesting that it has to be related to some kind of planar limit. These properties can be tested by familiar field theoretical perturbative computations on $\mathbb{R}_\theta$, using the star-product formalism: in [46, 52] Wilson loops on noncommutative plane were studied by perturbative methods. In particular the authors considered in [52] a k-winding circular contour, obtaining the complete $\mathcal{O}(g^4)$ result: it is not difficult to show that, in our limit, only the leading planar contribution survives in their expression, consistently reproducing eq. (6.17).

7. Conclusions and outlook

The TEK model naturally provides a non-perturbative definition of noncommutative Yang-Mills theory in every dimension, which can be employed, in principle, for concrete computer simulations. In two dimensions this approach becomes also a feasible tool for analytic computations due to the relative simplicity of the relevant matrix model. Previous investigations in the continuum theory offer a number of results to be checked and suggest the possibility to use the discrete formulation to unveil new properties. We think, here, we have made a couple of steps in these directions: on one hand we have proposed an unconstrained TEK model describing discretized noncommutative two dimensional Yang-Mills theory on a (p,q) projective module. On the other, we have been able to solve the related classical equations of motion; more importantly we have found a precise match between the solutions with finite action, in the relevant double scaling limit, and the critical points of the continuum theory. The classical solutions on a (p,q) projective module possess a rather non-trivial structure: we hope that having recovered it from the discretized formulation be a strong evidence of the correctness of our unconstrained approach. This encourages us to take a further step and to try to solve completely the quantum theory within this

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framework. In doing so, we could provide the natural derivation of the beautiful partition function proposed in [23] and open the possibility to compute general observables in a matrix model language. The fact the final theory should be semiclassical exact, as shown in [23], makes this attempt not completely hopeless, having in our case a complete understanding of the classical solutions and of their moduli space. The last section of the paper has been devoted to elucidate what are the new issues that arise in trying to tackle the computation, when considered in its full complexity: we hope to report in the near future about progresses in this direction.

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A. Morita equivalence on the fuzzy torus

The Morita equivalence is a surprising symmetry when seen from a conventional field theoretical point of view: it connects indeed theories with different gauge groups, different coupling constants and living on different tori. Physically it can be understood as coming from the T-duality possessed by the stringy ancestors of these models, while mathematically it expresses the fact that certain classes of algebras share the same representation theory.

In particular in two dimensions the Morita equivalence transformations are realized by a group element of $\mathrm{SL}(2,\mathbb{Z})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{A.1}$$

On the parameters, specifying the module, the transformation acts as

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{A.2}$$

In the commutative language, eqs. (A.2) corresponds to a non-trivial change of the gauge group and the Chern class of the original theory. The noncommutative parameters θ transforms with the Möbius transformation associated to eq. (A.1)

$$\theta' = \frac{a\,\theta + b}{c\,\theta + d}, \tag{A.3}$$

while the dimension of the module $\mathcal{E}$ scales as

$$\dim \mathcal{E}' = \frac{\dim \mathcal{E}}{|c\,\theta + d|}. \tag{A.4}$$

Finally the invariance of the noncommutative Yang-Mills action dictates the transformation rules for the remaining relevant objects

$$L' = |c\,\theta + d| L \qquad g'^2 = |c\,\theta + d| g^2 \qquad \Phi' = \Phi(c\,\theta + d)^2 - \frac{c(c\,\theta + d)}{2\pi R^2}. \tag{A.5}$$

In the following we will try to understand how this symmetry is realized in our description of the fuzzy torus. The first observation is that in our framework the integer N carries the information about the size $L = Na$ of the lattice and so its natural transformation under Morita is

$$N' = N(c\theta + d) = (2cr + dN)\,. \tag{A.6}$$

The transformation for θ eq. (A.3) can be now translated into a transformation rule for integer $2r$ appearing in our definition of the noncommutative lattice

$$2r' = (2ra + bN)\,. \tag{A.7}$$

That the lhs of the above equation is even must be understood modulo N'. Here we have limited ourselves to the transformations for which the combinations is $(2cr + dN)$ is positive. This is not really a restriction, because it is equivalent to state that our duality group is $\mathrm{PSL}(2, \mathbb{Z})$.

Now we stress that our final TEK model is completely defined by three parameters

$$\begin{aligned} n &= Np - 2rq\,, \\ m &= kq - ps\,, \\ \beta N &= \frac{2N^3}{g^2 A}\,, \end{aligned} \tag{A.8}$$

the last one being the total coefficient in front of the action. Let us see how they change under Morita. For the first one, we have

$$\begin{aligned} n' &= N'p' - 2r'q' = (2cr + dN)(ap + bq) - (2ra + bN)(cp + dq) \\ &= (2acrp + 2bcrq + adNp + bdNq - 2racp - 2rcbN - bcNp - bdNq) \\ &= (ad - bc)(Np - 2rq) = (Np - 2rq) = n\,. \end{aligned} \tag{A.9}$$

In other words, the parameter n is invariant under Morita equivalence. The transformations of the parameter m is more subtle, since its definition involves two new integers (k, s), which are implicitly defined by the Diophantine equation

$$(2r)s - kN = 1\,. \tag{A.10}$$

By imposing that s' and k' satisfy the same equation with N' and $2r'$, we obtain

$$s' = ck + ds \qquad k' = ak + bs\,. \tag{A.11}$$

Therefore, we have

$$\begin{aligned} m' &= k'q' - p's' = (ak + bs)(cp + dq) - (ck + ds)(ap + bq) \\ &= (ad - bc)(kq - ps) = kq - ps = m\,. \end{aligned} \tag{A.12}$$

Namely, also the parameter m is invariant under the Morita equivalence. A straightforward computation shows that also the last one is unaffected. Thus our model naturally encodes this symmetry, all his constituents being unchanged.

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B. The relation with the noncommutative Wilson lattice action

In section 2 we have tried to realize general projective modules at discretized level: the approach we have followed is similar, in the spirit, to the operatorial construction adopted by Connes and Rieffel in their seminal paper [34] on continuous noncommutative YM_2. Alternatively, as thoroughly reviewed by Konechny and Schwarz in [53], these modules can be also understood in terms of algebra of functions endowed with deformed products: the classical example is the Moyal product on Schwartz spaces. This second approach can be also adapted to the discrete formulation with the advantage that a "formal" space-time lattice structure will show up. However we have to stress that the emerging lattice will be only indirectly correlated to the original one: it will posses a different number of sites and a different θ. Technically we will obtain a discretized representation of the torus generated by the translations.[5]

Writing our reduced TEK model as a lattice-like Wilson action proceeds through a by now well-established series of steps [4, 11]. However, some attention has to be paid on initial set of geometrical data to be employed. The θ associated with our reduced TEK model is m/n and as in section 2 the consistency of the lattice construction would force m to be even. Actually, this is not a real limitation: in fact if we fix the geometrical data N, $2r$, p and q, the ambiguity intrinsic in the Diophantine equation determining m allows us to choose it even.

With this remark in mind one begins by defining the operator $\Delta(\vec{x})$ which realizes the mapping between operators and functions on the lattice. In terms of twist-eaters

$$\begin{aligned} Z_1 &= (\Gamma_1)^{kq-sp}\,, \\ Z_2 &= \Gamma_2\,, \end{aligned} \tag{B.1}$$

we have ($\vec{k} = (2\pi\frac{m_1}{na}, 2\pi\frac{m_2}{na})$)

$$\hat{\Delta}(\vec{x}) = \frac{1}{n^2}\sum_{m_{1,2}} \exp\left(i\vec{k}\cdot\vec{x})\exp(i\pi\frac{m\,m_1 m_2}{n}\right) Z_1^{m_1} Z_2^{m_2}\,. \tag{B.2}$$

The vector $\vec{x}$ "corresponds" to the point $(n_1 a, n_2 a)$ on a lattice of size n (as usual we denote a as the lattice spacing). Then to each operator $\hat{f}$ (matrix in our discretized case) we can associate the function

$$f(\vec{x}) = n\,\mathrm{Tr}\left[\hat{f}\hat{\Delta}(\vec{x})\right] \tag{B.3}$$

and conversely to each function $f(\vec{x})$ on the lattice

$$\hat{f} = \sum_{\vec{x}} f(\vec{x})\hat{\Delta}(\vec{x})\,. \tag{B.4}$$

[5] This point has been often overlooked in the literature. Also in the continuum version of the theory when written in the star-product formalism θ is not that one of the original torus describing instead the derivations algebra. Morita equivalence, nevertheless, connects these two tori.

That this mapping be an isomorphism of algebras naturally indicates how to deform the product between functions and in details we have

$$
\begin{aligned}
f(\vec{x}) \star g(\vec{x}) &= n \operatorname{Tr}\left[\hat{g}\hat{f}\hat{\Delta}(\vec{x})\right] = n \sum_{\vec{y},\vec{z}} f(\vec{z})g(\vec{y}) \operatorname{Tr}\left[\hat{\Delta}(\vec{y})\hat{\Delta}(\vec{z})\hat{\Delta}(\vec{x})\right] \\
&= \sum_{\vec{y},\vec{z}} f(\vec{z})g(\vec{y})\mathcal{K}(\vec{x}-\vec{y},\vec{x}-\vec{z})\,.
\end{aligned}
\tag{B.5}
$$

The definition of $\mathcal{K}(\vec{x}-\vec{y},\vec{x}-\vec{z})$ is given by

$$
\mathcal{K}(\vec{x}-\vec{y},\vec{x}-\vec{z}) = n \operatorname{Tr}\left[\hat{\Delta}(\vec{y})\hat{\Delta}(\vec{z})\hat{\Delta}(\vec{x})\right]
\tag{B.6}
$$

and in the following we shall compute the r.h.s. To reduce the amount of algebra is useful to define the combination

$$
u_{\vec{k}} = \exp\left(-i\pi\frac{mm_1m_2}{n}\right) Z_1^{m_1} Z_2^{m_2}\,,
\tag{B.7}
$$

which satisfies the simple composition rule

$$
u_{\vec{k}}u_{\vec{\ell}} = \exp(\pi i\Theta(k_1\ell_2 - k_2\ell_1))u_{\vec{k}+\vec{l}}
\tag{B.8}
$$

and the trace property

$$
\operatorname{Tr}(u_{\vec{k}}u_{\vec{\ell}}) = n\delta_{\vec{k}+\vec{l}}\,,
\tag{B.9}
$$

with $\Theta = m/n$. Now the r.h.s. of eq. (B.6) can be rewritten as

$$
\begin{aligned}
&= \frac{1}{n^5}\sum_{\vec{k},\vec{k}',\vec{\ell}} e^{i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{y}+i\vec{\ell}\cdot\vec{z}} \operatorname{Tr}(u_{-\vec{k}'}u_{-\vec{\ell}}\,u_{-\vec{k}}) \\
&= \frac{1}{n^5}\sum_{\vec{k},\vec{k}',\vec{\ell}} e^{i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{y}+i\vec{\ell}\cdot\vec{z}} e^{\pi i\Theta(k_1'k_2+k_1'\ell_2+k_2\ell_1-k_1k_2'-k_2'\ell_1-k_1\ell_2)} \operatorname{Tr}(u_{-\vec{k}'-\vec{\ell}-\vec{k}}) \\
&= \frac{1}{n^4}\sum_{\vec{k}',\vec{\ell}} e^{i\vec{k}'\cdot(\vec{y}-\vec{x})+i\vec{\ell}\cdot(\vec{z}-\vec{x})} e^{\pi i\Theta(k_1'\ell_2-k_2'\ell_1)} \\
&= \frac{1}{n^4}\sum_{\vec{k}',\vec{\ell}} e^{ik_1'(y_1-x_1+\pi\Theta\ell_2)+ik_2'(y_2-x_2-\pi\Theta\ell_1)+i\vec{\ell}\cdot(\vec{z}-\vec{x}))}\,.
\end{aligned}
\tag{B.10}
$$

The sum over the momentum $\vec{k}$ imposes that the two combinations,

$$
\frac{y_1 - x_1 + \pi\Theta\ell_2}{a} = \frac{1}{a}\left(m_1a - n_1a + \frac{2\pi^2}{na}j_2\frac{nm}{4\pi^2}a^2\right) = \left(m_1 - n_1 + \frac{m}{2}j_2\right)
\tag{B.11}
$$

and

$$
\frac{y_2 - x_2 - \pi\Theta\ell_1}{a} = \frac{1}{a}\left(m_2a - n_2a - \frac{2\pi^2}{na}j_1\frac{nm}{4\pi^2}a^2\right) = \left(m_2 - n_2 - \frac{m}{2}j_1\right),
\tag{B.12}
$$

vanish modulo n. Notice that since $\vec{j} = \dfrac{na\vec{\ell}}{2\pi}$ is an integer vector, we must require that

$$
j_1 = \frac{2(n_2 - m_2 + h_1n)}{m} \qquad \text{and} \qquad j_2 = \frac{2(m_1 - n_1 + h_2n)}{m}
\tag{B.13}
$$

belong to $\mathbb{Z}$ for a suitable choice of $\vec{h}$. Recall also that m can be always taken even as stressed before. These conditions can be always met: in fact, the equations $mj_i/2 - nh_i =$ (a given integer) are always solvable when n and $m/2$ are coprime. Denoting, with s' and k' the solution of the Diophantine equation

$$s'm - k'n = 1\,, \tag{B.14}$$

the form of j_i and h_i is given by

$$j_1 = 2s'(n_2 - m_2) \qquad h_1 = k'(n_2 - m_2) \quad \Rightarrow \quad \ell_1 = -\frac{4\pi s'}{na^2}(y_2 - x_2) \tag{B.15}$$

and

$$j_2 = 2s'(m_1 - n_1) \qquad h_1 = k'(m_1 - n_1) \quad \Rightarrow \quad \ell_2 = \frac{4\pi s'}{na^2}(y_1 - x_1)\,. \tag{B.16}$$

Substituting this result in the sum we finally have

$$\mathcal{K}(\vec{y} - \vec{x}, \vec{z} - \vec{x}) = \frac{1}{n^2} \exp \frac{4\pi i s'}{na^2}\Big[(y_1 - x_1)(z_2 - x_2) - (y_2 - x_2)(z_1 - x_1)\Big]\,. \tag{B.17}$$

We can easily see how the star-product is explicitly realized: let us consider the basis for the functions on the toroidal lattice given by the $u_{\vec{k}}(\vec{x}) = \exp(i\vec{k}\cdot\vec{x})$ as defined in section 2. Their algebra is now modified by the star product:

$$\begin{aligned} u_{\vec{k}}(\vec{x}) \star u_{\vec{k}'}(\vec{x}) &= \sum_{\vec{y},\vec{z}} u_{\vec{k}}(\vec{z}) u_{\vec{k}'}(\vec{y}) \mathcal{K}(\vec{x} - \vec{y}, \vec{x} - \vec{z}) \\ &= \frac{1}{n^2} \sum_{\vec{y},\vec{z}} e^{\frac{4\pi i s'}{na^2}[(y_1 - x_1)(z_2 - x_2) - (y_2 - x_2)(z_1 - x_1)] + i\vec{k}\cdot\vec{z} + i\vec{k}'\cdot\vec{y}} \\ &= \frac{1}{n^2} \sum_{\vec{y},\vec{z}} e^{iy_1[\frac{4\pi s'}{na^2}(z_2 - x_2) + k'_1] - iy_2[\frac{4\pi s'}{na^2}(z_1 - x_1) - k'_2] + \frac{4\pi i s'}{na^2}(z_1 x_2 - z_2 x_1) + i\vec{k}\cdot\vec{z}}\,. \end{aligned} \tag{B.18}$$

To evaluate this expression, we note that the sum over $\vec{y}$ is zero unless the following combinations vanish

$$\frac{4\pi s'}{na^2}(z_2 - x_2) + k'_1 = \frac{2\pi}{na}(2s'(m_2 - n_2) + j_1) = 0 \tag{B.19}$$

and

$$\frac{4\pi s'}{na^2}(z_1 - x_1) - k'_2 = \frac{2\pi}{na}(2s'(m_1 - n_1) - j_2) = 0\,. \tag{B.20}$$

Here we have set $\vec{x} = \vec{n}a$, $\vec{z} = \vec{m}a$ and $\vec{k}' = 2\pi\vec{j}/na$. Taking into account the periodicity in n, the integer solutions of the above equation are the solutions of the Diophantine equations

$$2(m_2 - n_2)s' - h_1 n = -j_1 \qquad \text{and} \qquad 2(m_1 - n_1)s' - h_2 N = j_2\,. \tag{B.21}$$

Comparing this equation with eq. (B.14), we obtain the solutions

$$\begin{aligned} m_2 &= -j_1\frac{m}{2} + n_2 \quad \Rightarrow \quad z_2 = x_2 - \frac{na^2 m}{4\pi}k'_1 \\ h_1 &= -kj_1 \\ m_1 &= j_2\frac{m}{2} + n_1 \quad \Rightarrow \quad z_1 = x_1 + \frac{na^2 m}{4\pi}k'_2 \\ h_2 &= kj_2\,. \end{aligned} \tag{B.22}$$

Then the product of two Block-waves is

$$
\begin{aligned}
&= e^{\frac{4\pi i s'}{na^2}\left(\left(x_1+\frac{na^2m}{4\pi}k'_2\right)x_2-\left(x_2-\frac{na^2m}{4\pi}k'_1\right)x_1\right)+ik_1\left(x_1+\frac{na^2m}{4\pi}k'_2\right)+ik_2\left(x_2-\frac{na^2m}{4\pi}k'_1\right)} \\
&= e^{i(\vec{k}+\vec{k}')\cdot\vec{x}+\pi i\left(\frac{na^2m}{4\pi^2}\right)(k_1k'_2-k_2k'_1)} = e^{\pi i\Theta(k_1k'_2-k_2k'_1)}u_{\vec{k}+\vec{k}'}(\vec{x})\,. \qquad \text{(B.23)}
\end{aligned}
$$

Having obtained an explicit discretized form of the star-product, we can construct a Wilson-like action: following [11], it is obtained through some algebraic manipulations. We rewrite

$$
\begin{aligned}
S_{TEK} = \frac{\beta N}{n^2}\sum_{\vec{x}}\mathrm{Tr}\Bigg[&\left(\exp\left(\frac{\pi i m}{n}\right)V_1V_2-\exp\left(-\frac{\pi i m}{n}\right)V_2V_1\right)\times \\
&\times\left(\exp\left(-\frac{\pi i m}{n}\right)V_2^\dagger V_1^\dagger-\exp\left(\frac{\pi i m}{n}\right)V_1^\dagger V_2^\dagger\right)\hat{\Delta}(\vec{x})\Bigg]\,, \qquad \text{(B.24)}
\end{aligned}
$$

using the new variables $V_i = \hat{V}_i\Gamma_i$ as

$$
\begin{aligned}
S_{TEK} &= 2\beta N n-\beta\frac{N}{n}\sum_x\sum_{i\neq j}\hat{\mathcal{V}}_i(\vec{x})\star\hat{\mathcal{V}}_j(\vec{x}+a\vec{i})\star\hat{\mathcal{V}}_i^*(\vec{x}+a\vec{j})\star\hat{\mathcal{V}}_j^*(\vec{x}) \\
&= \frac{1}{p-\theta q}\left(2\beta n^2-\beta\sum_x\sum_{i\neq j}\hat{\mathcal{V}}_i(\vec{x})\star\hat{\mathcal{V}}_j(\vec{x}+a\vec{i})\star\hat{\mathcal{V}}_i^*(\vec{x}+a\vec{j})\star\hat{\mathcal{V}}_j^*(\vec{x})\right) \qquad \text{(B.25)}
\end{aligned}
$$

where the explicit form of the discretized star-product eqs. (B.5) and (B.17) has been taken into account. This is the promised Wilson-like representation of our TEK model, improved by a zero-point subtraction, and expressed just in terms of geometrical data (p,q), θ, n and a. We must stress few facts: first of all the dimensions of the lattice is n^2 and not N^2. This is expected: in fact, as noted in [11], the dimension of the lattice is related to the representation of the translation operators. Accordingly the star-product realizes a torus with $\theta = m/n$ and not $\theta = 2r/N$, which is the noncommutative parameter of a derivative torus: this again is in complete agreement with [11], where the Morita dual parameter appears.

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